Continuum Dynamics

Contents

- mathematical fundamentals
  - → tensor calculus
  - → functional analysis
- derivation of conservation laws
- Newtonian fluids
- Navier-Stokes equations
- qualitative properties of NS eqs. (existence of solution of the incompressible flow problem)

MATH TOOLS

- $\mathbb{R}^n$ space $(\overset{\text{typically}}{n=3})$, $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$ standard basis
- standard scalar (dot) product $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$
- induced (Euclidean) norm $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$

What is a tensor

V... vector space over $\mathbb{T}$ $(\mathbb{T} = \mathbb{R})$, $\dim V = n$. A (p,q)-type tensor of order $p+q$ where $p, q \in \mathbb{N}$ is a multilinear form

$$\mathbb{T} : \underbrace{V^* \times V^* \times V^* \cdots \times V^*}_{p\text{-times}} \times \underbrace{V \times V \times \dots \times V}_{q\text{-times}} \to \mathbb{R}$$

V^*.. dual space

resp. $\quad \mathbb{T} \in \underbrace{V \otimes V \otimes \cdots \otimes V}_{p\text{-times}} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{q\text{-times}}$

$\otimes$ ... tensor product of vector spaces

Example: $V_1 \otimes V_2$ with $\dim V_1 = n_1$, $\dim V_2 = n_2$

dim $= n_1 \cdot n_2$

$V_1 = \mathbb{R}^{n_1}$
$V_2 = \mathbb{R}^{n_2}$ with standard bases $\vec{e}_1 \dots \vec{e}_{n_1}$, $\vec{f}_1 \dots \vec{f}_{n_2}$
$V_1 \otimes V_2 = \mathbb{R}^{n_1 n_2}$

then the basis of $V_1 \otimes V_2$ has the elements $\vec{e}_i \otimes \vec{f}_j =:$

$(\vec{g}_{ij})_{kl} = e_i^k f_j^l$

Pr: type $(1,0)$ tensor $\quad \vec{v} : V^* \to \mathbb{R}$, $\vec{v} \in V$ $\quad \updownarrow$ Riesz ... it's a vector

note: $\vec{v} \cdot \vec{h} = \overleftarrow{v}(\vec{h})$

tensor type $(0,1)$ $\quad \overleftarrow{v} : V \to \mathbb{R}$, $\overleftarrow{v} \in V^*$ $\quad \updownarrow$ it's a linear functional

Representation of tensor $\mathbb{T}$ in a basis

$$[\mathbb{T}]_\chi = \hat{\mathbb{T}} = [\mathbb{T}] = \left(\tau^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\right)$$

if $q = 0 \Rightarrow \mathbb{T}$
$p = 0 \Rightarrow \mathbb{T}$
$pq \neq 0 \Rightarrow \mathbb{T}$

is called contravariant / covariant / mixed type tensor

Einstein summation rule

$$\rho^{i}_{klm} = \sigma^{ij}_{kl}\,\tau_{jm} = \sum_{j=1}^{n} \sigma^{ij}_{kl}\,\tau_{jm}$$

$$\underset{\hat{R}}{[R]} \qquad \underset{\hat{S}}{[S]} \qquad \underset{\hat{T}}{[T]}$$

Example: tensor $\mathbb{T}$ of type $(1,2)$ is applied to $(\underleftarrow{u}, \vec{v}, \vec{w}) \in V^* \times V^2$

$$\mathbb{T}(\underleftarrow{u}, \vec{v}, \vec{w}) = \tau^{i}_{jk}\, u_i\, v^j w^k \qquad \left(\sum_i^n \sum_j^n \sum_k^n\right)$$

$\vec{v} = v^i \vec{x}_i$

$\underleftarrow{u} = u_i \underleftarrow{x}^i$

$(\vec{x}_1, \dots, \vec{x}_n) = \mathcal{X}$ basis of V

$(\underleftarrow{x}^1, \dots, \underleftarrow{x}^n) = \mathcal{X}^*$ basis of V^* (dual basis)

$\underleftarrow{x}^i(\vec{v}) = v^i$ "i-th coordinate functional"

TENSOR PRODUCT & INNER PRODUCT

$\hat{S} = [S] = \left(\sigma^{kl}_{m}\right)$ order 3, type (2,1) $\qquad \hat{T} = [T] = \left(\tau^{i}_{j}\right)$ order 2, type (1,2)

TENSOR PROD.

$$U = S \otimes T \qquad [U] = \left(\mu^{ikl}_{jm}\right) \text{ where } \mu^{ikl}_{jm} = \sigma^{kl}_{m} \cdot \tau^{i}_{j}$$

order 3+2

"normal" multiplication of numbers

INNER (scalar) PROD.

$[S] = \left(\sigma^{kl}_{m}\right)$ order 3 $\qquad [T] = \left(\tau^{m}_{kl}\right)$ order 3

$$u = S \odot T = \sigma^{kl}_{m}\,\tau^{m}_{kl}$$

COVARIANT & CONTRAVARIANT TENSORS

V over $\mathbb{R}$, $\mathcal{X} = (\vec{x}_1, \dots, \vec{x}_n)$, $\mathcal{Y} = (\vec{y}_1, \dots, \vec{y}_n)$ are two bases of V

Let $\vec{v} \in V$. Then

j-th coordinate functional

$$\vec{v} = \sum_{j=1}^{n} x^j(\vec{v})\vec{x}_j = \sum_{i=1}^{n} y^i(\vec{v})\vec{y}_i$$

What is the relationship of coordinates of $\vec{v}$ in bases $\mathcal{X}$ and $\mathcal{Y}$?

$$y^i(\vec{v}) = y^i\left(\sum_{j=1}^{n} x^j(\vec{v})\vec{x}_j\right) = \sum_{j=1}^{n} x^j(\vec{v})\, y^i(\vec{x}_j) = \sum_{j=1}^{n} y^i(\vec{x}_j)\, x^j(\vec{v})$$

$$\Rightarrow \boxed{\ulcorner \vec{v} \urcorner_{\mathcal{Y}} = {}^{\mathcal{X}}\hat{P}^{\mathcal{Y}} \ulcorner \vec{v} \urcorner_{\mathcal{X}}} \quad (*) \quad \text{where} \quad {}^{\mathcal{X}}\hat{P}^{\mathcal{Y}}_{ij} = \vec{y}^{\,i}(\vec{x}_j)$$

analogously, $\ulcorner \vec{v} \urcorner_{\mathcal{X}} = {}^{\mathcal{Y}}\hat{P}^{\mathcal{X}} \ulcorner \vec{v} \urcorner_{\mathcal{Y}}$ where ${}^{\mathcal{Y}}\hat{P}^{\mathcal{X}}_{ij} = \vec{x}^{\,i}(\vec{y}_j)$

and obviously ${}^{\mathcal{Y}}\hat{P}^{\mathcal{X}} = ({}^{\mathcal{X}}\hat{P}^{\mathcal{Y}})^{-1}$ (as $\ulcorner \vec{v} \urcorner_{\mathcal{X}} = {}^{\mathcal{X}}\hat{P}^{\mathcal{Y}} \ulcorner \vec{v} \urcorner_{\mathcal{Y}} = {}^{\mathcal{X}}\hat{P}^{\mathcal{Y}}\, {}^{\mathcal{Y}}\hat{P}^{\mathcal{X}} \ulcorner \vec{v} \urcorner_{\mathcal{X}}$)

In particular, if $V = \mathbb{R}^n \Rightarrow \mathbb{X} = (\vec{x}_1 \dots \vec{x}_n)$ $\mathbb{Y} = (\vec{y}_1 \dots \vec{y}_n)$

Let $\mathbb{Y} = \mathbb{X}\mathbb{A}$ where $\mathbb{A} \in \mathbb{R}^{n \times n}$ is a regular matrix

Then $\vec{v} = \mathbb{Y} \ulcorner \vec{v} \urcorner_{\mathcal{Y}} = \underline{\mathbb{X}\mathbb{A}\ulcorner \vec{v} \urcorner_{\mathcal{Y}}} = \underline{\mathbb{X}\ulcorner \vec{v} \urcorner_{\mathcal{X}}}$

$$\ulcorner \vec{v} \urcorner_{\mathcal{Y}} = \mathbb{A}^{-1} \ulcorner \vec{v} \urcorner_{\mathcal{X}} \quad (**)$$

$(*) + (**) \Rightarrow \mathbb{A} = {}^{\mathcal{Y}}\hat{P}^{\mathcal{X}} = ({}^{\mathcal{X}}\hat{P}^{\mathcal{Y}})^{-1}$ coordinates of $\vec{v}$ transform "against" (inversely to) the transformation of basis $\Rightarrow$ CONTRA-variant

Note : $(\underleftarrow{x}^1, \dots \underleftarrow{x}^n)$ is the basis of V^* $\quad (\underleftarrow{x}^i(\vec{x}_j) = \delta_{ij}$ bi-orthogonality property)

Now, let's take $\underleftarrow{w} \in V^*$. Then for $\vec{v} \in V$, we have

$$\underleftarrow{w}(\vec{v}) = \underleftarrow{w}\left(\sum_{i=1}^{n} \underleftarrow{x}^i(\vec{v})\vec{x}_i\right) = \sum_{i=1}^{n} \underleftarrow{x}^i(\vec{v})\underleftarrow{w}(\vec{x}_i) = \Big(\underbrace{\sum \underleftarrow{w}(\vec{x}_i)\underleftarrow{x}^i}_{=\underleftarrow{w}}\Big)(\vec{v})$$

i.e. the coordinates of $\underleftarrow{w}$ in basis $\mathcal{X}^* = (\underleftarrow{x}^1, \dots, \underleftarrow{x}^n)$
are $\underleftarrow{w}(\vec{x}_i)$ $\quad i = 1, \dots, n$.

Now : What is the relationship of coordinates of $\underleftarrow{w}$ in bases $\mathcal{X}^*$ and $\mathcal{Y}^*$?

$$\underleftarrow{w}(\vec{y}_i) = \underleftarrow{w}\left(\sum_{j=1}^{n} \underleftarrow{x}^j(\vec{y}_i)\vec{x}_j\right) = \sum_{j=1}^{n} \underleftarrow{x}^j(\vec{y}_i)\underleftarrow{w}(\vec{x}_j)$$

in vector notation :

$$[\underleftarrow{w}]_{\mathcal{Y}^*} = ({}^{\mathcal{Y}}\hat{\mathbb{P}}^{\mathcal{X}})^T [\underleftarrow{w}]_{\mathcal{X}^*}$$

$$\boxed{\text{recall: } \; {}^{\mathcal{Y}}\hat{\mathbb{P}}^{\mathcal{X}}_{ij} = \underleftarrow{x}^i(\vec{y}_j)}$$

this is equal to $\mathbb{A}^T$ if $V = \mathbb{R}^n$

Coordinates of $\underleftarrow{w}$ transform "with" (proportionally to) the basis
= co-variant

→ elements of V^* are called co-vectors (represented by means of Riesz theorem by the elements of V)

NOTE: in Physics :

(contravariant) vectors : position, velocity, force .. [...m]

co-vectors : ∇f
$$[\nabla f] = \frac{1}{m}$$

"OG"

ORTHOGONAL TRANSFORMATIONS OF TENSORS

ON = orthonormal

$\mathbb{R}^3$ with an ON basis $\mathcal{X} = (\vec{x}_1, \vec{x}_2, \vec{x}_3)$ + std. scalar product

Then ($\vec{v} \in \mathbb{R}^3$) $\vec{x}_j \cdot \vec{v} = \vec{x}_j \cdot \left(x^i(\vec{v})\, \vec{x}_i \right) = x^i(\vec{v}) \underbrace{(\vec{x}_j \cdot \vec{x}_i)}_{\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}} = x^j(\vec{v})$

$\vec{v} = (\vec{x}_j \cdot \vec{v})\, \vec{x}_j$

if $\mathcal{Y}$ is an orthonormal basis satisfying

$$(\vec{y}_1\ \vec{y}_2\ \vec{y}_3) = (\vec{x}_1\ \vec{x}_2\ \vec{x}_3)\, \mathbb{A} \quad \text{where } \mathbb{A} \text{ is OG}$$

$$\Leftrightarrow \quad \underline{\underline{\mathbb{A}^{-1} = \mathbb{A}^T}}$$

If $v \in V^*$ is represented by $\vec{v} \in V$: $v(\vec{h}) = \vec{v} \cdot \vec{h}$

then $\lceil v \rceil_{\mathcal{X}^*} = \lceil \vec{v} \rceil_{\mathcal{X}}$

$\Rightarrow$ $\boxed{\text{contra-variant} = \text{covariant}}$ $\Rightarrow$ all indices will be written as Lower indices

- orthogonal transformation of tensors $\qquad \mathbb{A} = (\alpha_{ij})$

$$\lceil \mathbb{T} \rceil_{\mathcal{X}} \; (= \hat{\mathbb{T}}) = (\tau_{i_1 \ldots i_s}) \;, \quad \lceil \mathbb{T} \rceil_{\mathcal{Y}} = (\tau'_{i_1 \ldots i_s})$$

where $\tau'_{i_1 \ldots i_s} = \alpha_{i_1 j_1} \alpha_{i_2 j_2} \cdots \alpha_{i_s j_s} \tau_{j_1 \ldots j_s}$.

vector: $v'_i = \alpha_{ij} v_j$

$\tau'_{ij} = \alpha_{iI} \alpha_{jJ} \tau_{IJ}$

In this fluid dynamics course, we'll use:

- tensors of 2nd order (= matrices) + ON bases only

representation of $\mathbb{T}$ in $\mathcal{X}$ (usually $\mathcal{E}$) is identified with $\mathbb{T}$

$\mathbb{T} \equiv \hat{\mathbb{T}} = \lceil \mathbb{T} \rceil_{\mathcal{X}}$ $\qquad \Rightarrow$ "^" not used anymore $\qquad \lceil \mathbb{T} \rceil_{\mathcal{Y}} = \mathbb{A}^T \lceil \mathbb{T} \rceil_{\mathcal{X}} \mathbb{A}$

$\mathbb{T}' \equiv \hat{\mathbb{T}}' = \lceil \mathbb{T} \rceil_{\mathcal{Y}}$ $\qquad \mathbb{T}' = \mathbb{A}^T \mathbb{T} \mathbb{A}$

- $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

Kroneckerovo δ

- $\varepsilon_{ijk} = \begin{cases} 1 & \dots (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \dots \text{ — '' — odd — '' —} \\ 0 & \end{cases}$

Levi-Civita symbol

in $\mathbb{R}^3$

- $(\vec{a} \times \vec{b}) = (\varepsilon_{ijk} a_j b_k) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- transpose $\mathbb{T} = (\tau_{ij})$ i.e. $\mathbb{T}^T = (\tau_{ji})$

  (Note: $\vec{y}\cdot\mathbb{T}\vec{x} = \vec{x}\cdot\mathbb{T}^T\vec{y} \quad \forall \vec{x},\vec{y}$)

- $\mathbb{E}$ is symmetric $\Leftrightarrow \mathbb{E} = \mathbb{E}^T$
- $\mathbb{W}$ is skew-symm. $\Leftrightarrow \mathbb{W} = -\mathbb{W}^T$

$\Big\}\Rightarrow \mathbb{E}\cdot\mathbb{W} = \Theta$

- scalar (inner) product of tensors : $\mathbb{S} \odot \mathbb{T} = \mathbb{S}\cdot\mathbb{T} = \sigma_{ij}\tau_{ij}$

- scalar product of tensor & vector : $\vec{v}\cdot\mathbb{T} = \mathbb{T}\cdot\vec{v} = \mathbb{T}\vec{v} = (\tau_{ij} v_j)$

- decomposition of any tensor $\mathbb{T} = \underbrace{\tfrac{1}{2}(\mathbb{T}+\mathbb{T}^T)}_{\substack{\mathbb{E} \\ \text{(sym.)}}} + \underbrace{\tfrac{1}{2}(\mathbb{T}-\mathbb{T}^T)}_{\substack{\mathbb{W} \\ \text{(skew-sym.)}}}$

- $\mathbb{W} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \qquad \vec{w} = \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = (w_i)$

$\Rightarrow (\forall \vec{v} \in \mathbb{R}^2)(\mathbb{W}\cdot\vec{v} = \vec{w}\times\vec{v})$

FROM NOW ON, ALL INDICES ARE <u>DOWN</u>!

SOME LINEAR ALGEBRA

- Let $A \in \mathbb{R}^{3\times 3}$, $A = (\alpha_{ij})$

then: $\det A = \varepsilon_{ijk}\, \alpha_{1i}\alpha_{2j}\alpha_{3k} = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{3} \alpha_{i\pi(i)}$

$$= \boxed{\frac{1}{3!}\varepsilon_{IJK}\, \varepsilon_{ijk}\, \alpha_{Ii}\alpha_{Jj}\alpha_{Kk}}$$

- <u>Cramer's rule</u> $\quad A\vec{x} = \vec{b}$, $A \in \mathbb{R}^{3\times 3}$ (regular)

$\iff x_i = \frac{1}{\det A}\Delta_i$ where Δ_i is the determinant of a matrix created from A by replacing its i'th column by $\vec{b}$

Corollary

- $AA^{-1} = I$. Denoting $A^{-1} = (\tilde{\alpha}_{ij})$, we have

$$\boxed{\tilde{\alpha}_{ij} = \frac{1}{\det A}\cdot \Delta_{ji}}$$

where Δ_{ji} is the determinant of a matrix created from A by replacing its i-th column by the j-th column of $\mathbb{I}$ $(= \vec{e}_j)$

Definition $\Delta_{ij} = (-1)^{i+j} \det A(i,j) \ldots$ cofactor of α_{ij}

matrix A, but with i-th row & j-th column removed

$$\begin{array}{cc} & i \\ I & \begin{pmatrix} \cdot & 0 & \cdot \\ - & 1 & - \\ \cdot & 0 & \cdot \end{pmatrix} \end{array}$$

i.e. $\Delta_{Ii} = (-1)^{I+i} \det A(I,i)$

$$\Rightarrow \boxed{\Delta_{Ii} = \frac{1}{2!}\varepsilon_{IJK}\, \varepsilon_{ijk}\, \alpha_{Jj}\alpha_{Kk}}$$

Differential calculus with vectors & tensors in $\mathbb{R}^3$

nabla
hel

- $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)^T = (\partial_1, \partial_2, \partial_3)^T = (\partial_i)$ $\longleftarrow$ for functions of time & space, ∂_t is omitted!

gradient of scalar field
- $(\nabla f = \partial_i f)$

gradient of vector field
- $\nabla \vec{f} = (\partial_i f_j) = (\nabla \otimes \vec{f})^T$

divergence of vector field
- $\nabla \cdot \vec{f} = (\partial_i f_i) = \partial_i f_i = \operatorname{div} \vec{f}$ $\begin{cases} >0 & \dots \text{ source} \\ <0 & \dots \text{ sink} \end{cases}$

divergence of order-2 tensor field
- $\nabla \cdot \mathbb{T} = (\partial_j \tau_{ij}) = \begin{pmatrix} \nabla \cdot (\tau_{11}\ \tau_{12}\ \tau_{13}) \\ \nabla \cdot (\tau_{21}\ \tau_{22}\ \tau_{23}) \\ \nabla \cdot (\tau_{31}\ \tau_{32}\ \tau_{33}) \end{pmatrix}$

- curl of a vector field

$\underline{\text{rot}}\, \vec{f} = \underline{\text{curl}}\, \vec{f} = \nabla \times \vec{f} = \varepsilon_{ikl}\, \partial_k f_l$

CZ / GER   EN

hole $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$(a \times b)_i = \varepsilon_{ikl}\, a_k b_l$

INVARIANTS OF A TENSOR

$\mathbb{T}' = (\tau'_{ij})$

- a scalar function $\lambda(\mathbb{T})$ is an invariant of $\mathbb{T}$ $\Leftrightarrow$

$$\lambda((\tau_{ij})) = \lambda((\tau'_{ij})) \quad \boxed{\begin{array}{l} Q = (q_{ij}) \;..\; \text{orthogonal} \\ \tau'_{ij} = q_{iI}q_{jJ}\tau_{IJ} \end{array}}$$

- For example, let $\mathbb{S} = (\sigma_{ij})$, $\mathbb{T} = (\tau_{ij})$

then: $\mathbb{S}\cdot\mathbb{T} = \mathbb{S}\odot\mathbb{T} = \sigma_{ij}\tau_{ij}$ is an invariant of both $\mathbb{S}$ and $\mathbb{T}$:

$$\boxed{\sigma'_{ij}\tau'_{ij}} = q_{ik}q_{jl}\sigma_{kl}\,q_{ir}q_{js}\tau_{rs} = \delta_{kr}\delta_{ls}\sigma_{kl}\tau_{rs} = \boxed{\sigma_{kl}\tau_{kl}}$$

δ_{kr} δ_{ls}

if in particular $\mathbb{S} = \mathbb{I}$: $\mathbb{I}\cdot\mathbb{T} = \delta_{ij}\tau_{ij} = \tau_{ii} = \mathrm{Tr}\,\mathbb{T}$

trace of $\mathbb{T}$

$$\ell(\lambda) = \det(\mathbb{T} - \lambda\mathbb{I})$$

as $\det(\mathbb{A}\mathbb{B}) = \det\mathbb{A}\cdot\det\mathbb{B}$, we have

$$\det(\underbrace{Q^T\mathbb{T}Q}_{\mathbb{T}'} - \lambda\mathbb{I}) = \det\left(Q^T(\mathbb{T}-\lambda\mathbb{I})Q\right) = \det(\mathbb{T}-\lambda\mathbb{I})$$

coefficients of the characteristic polynomial of $\mathbb{T}\in\mathbb{R}^{3\times3}$ are

the PRINCIPAL INVARIANTS of $\mathbb{T}$

$$\begin{cases} {}_{\mathbb{T}}I_1 = \mathrm{Tr}\,\mathbb{T} \\ {}_{\mathbb{T}}I_2 = \frac{1}{2}\left((\mathrm{Tr}\,\mathbb{T})^2 - \mathrm{Tr}(\mathbb{T}^2)\right) \\ {}_{\mathbb{T}}I_2 = \det\mathbb{T} \end{cases}$$

The Cayley–Hamilton theorem: $\mathbb{T}$ solves its own characteristic equation, i.e. $\ell(\mathbb{T}) = \mathbb{0}$

ISOTROPIC TENSOR-VALUED FUNCTIONS

A tensor-valued function $\mathbb{F} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is called ISOTROPIC

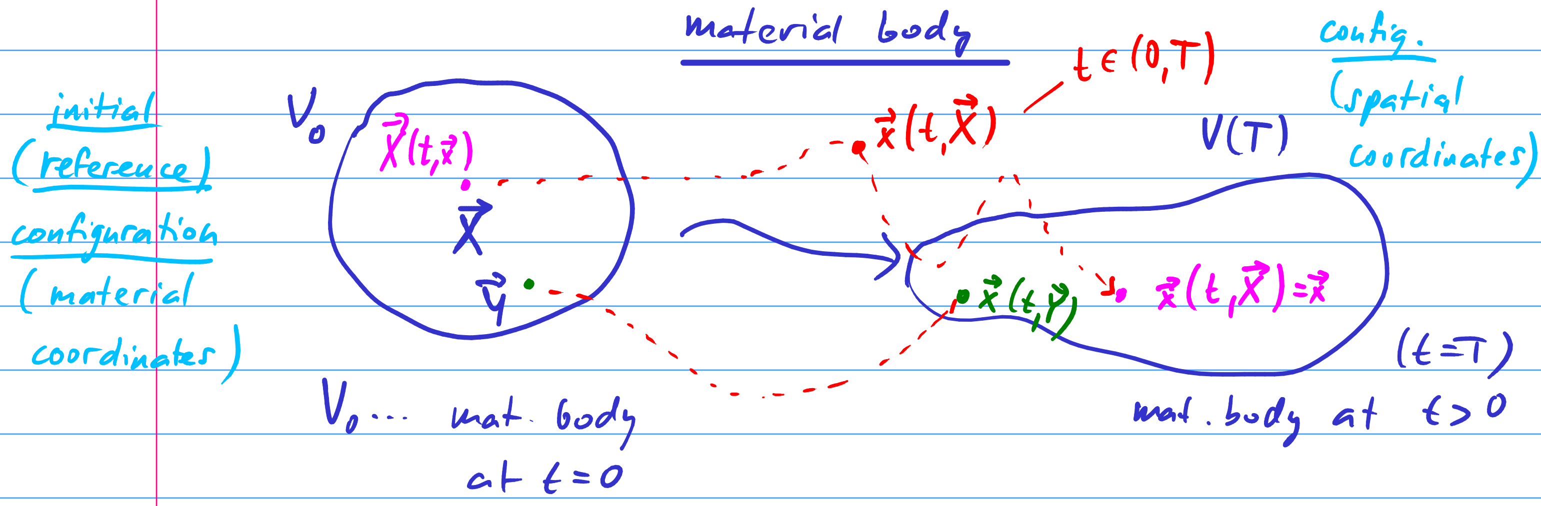
$\Leftrightarrow$ for each orthogonal transformation $\mathbb{Q}$ and each $\mathbb{T} \in \mathbb{R}^{3\times3}$

$$\mathbb{Q}^T \mathbb{F}(\mathbb{T}) \mathbb{Q} = \mathbb{F}(\mathbb{Q}^T \mathbb{T} \mathbb{Q}) \quad \text{holds}$$

or equivalently: $\underline{\mathbb{F}'(\mathbb{T}) = \mathbb{F}(\mathbb{T}')}$

KINEMATICS OF FLUIDS

$\rightarrow$ $V_0 \subset \mathbb{R}^3$ ... domain



we have a map $\vec{x} : \underbrace{(0,T)}_{\text{time interval}} \times V_0 \rightarrow \mathbb{R}^3$ position of $\vec{X}$ at time t

$\vec{x}$

$$\vec{x} = \vec{x}(t,\vec{X})$$
$$\vec{y} = \vec{x}(t,\vec{Y})$$

NOTE: $\dfrac{\partial \vec{x}}{\partial t}(t,\vec{X}) = \vec{v}(t,\vec{X})$ .. velocity of the point $\vec{X}$ at t

- $\vec{x}$ is regular and bijective, i.e. $\exists\, \vec{x}^{-1} = \vec{X}$

$$\vec{X} = \vec{X}(t,\vec{x})$$

and $\det \mathcal{J}_{\vec{x}} = \det\left(\dfrac{\partial x_i}{\partial X_j}\right) \neq 0$

$$\mathcal{J}_{\vec{x}} = \mathbb{F}(t,\vec{X}) = \left(\frac{\partial x_i}{\partial X_j}(t,\vec{X})\right)$$ deformation gradient

MATERIAL (SUBSTANSTIAL) DERIVATIVE

let $w : (0,T) \times V_0 \to \mathbb{R}$

and let $W : (0,T) \times \mathbb{R}^3 \to \mathbb{R}$ such that

w, W describe the same quantity in different coordinate systems

$$w(t,\vec{X}) = W(t,\vec{x}(t,\vec{X})) \Leftrightarrow W(t,\vec{x}) = w(t,\underbrace{\vec{X}}_{\vec{x}^{-1}}(t,\vec{x}))$$

$$= W(t,\vec{x}) \qquad\qquad = w(t,\vec{X})$$

here $\underset{\text{variable}}{\vec{x}} = \underset{\text{map}}{\vec{x}}(t,\vec{X})$, χ

$$\frac{\partial w(t,\vec{X})}{\partial t} = \frac{\partial}{\partial t}\left(W(\vec{\Phi}(t,\vec{X}))\right) =$$

where $\vec{\Phi}(t,\vec{X}) = \begin{pmatrix}\phi_0\\ \phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} t \\ \vec{x}(t,\vec{X})\end{pmatrix}$

$$\neq \frac{\partial W(t,\vec{x})}{\partial t}$$

$$= \sum_{k=1}^{3} \frac{\partial W}{\partial \phi_k}(\vec{\Phi}(t,\vec{X})) \cdot \frac{\partial \vec{\Phi}_k}{\partial t}(t,\vec{X}) =$$

$$= \frac{\partial W}{\partial t}(t,\vec{x}(t,\vec{X})) + \frac{\partial W}{\partial x_k}(t,\vec{x}(t,\vec{X})) \cdot \frac{\partial x_k}{\partial t}(t,\vec{X})$$

$$v_k(t,\vec{X}) = V_k(t,\vec{x}(t,\vec{X}))$$

$$= \frac{\partial W}{\partial t} + \vec{V} \cdot \operatorname{grad} W \Big|_{(t,\vec{x}(t,\vec{X})) = (t,\vec{x})}$$

$$= \underbrace{\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)}_{\frac{D}{Dt}} W(t,\vec{x}) = \frac{DW}{Dt}(t,\vec{x})$$

material derivative

NOTE: For vector quantities, $\frac{D}{Dt}$ is defined component-wise.

(Pr) acceleration of $\vec{X}$ at time t

$$\frac{\partial \vec{v}}{\partial t}(t,\vec{X}) = \frac{D\vec{V}}{Dt}(t,\vec{x}) \quad \text{where} \quad \vec{x} = \vec{x}(t,\vec{X})$$

CONTINUUM HYPOTHESIS

existuje míra M na $\mathbb{R}^3$

DEF: Material body V_0 is considered to be a continuum $\Leftrightarrow$ there exists a measure M on R with the physical meaning of mass (weight) which is continuous with respect to the classical Lebesgue measure m_3 (the volume), i.e.

$$m_3(A) = 0 \Rightarrow M(A) = 0$$

for all $A \subset V$.

Def: Let $\vec{\Psi}$ be a (vector) measure on V_0 such that for each $\vec{x} \in V_0$, there exists the limit

$$\vec{\psi}(\vec{x}) = \lim_{R\to 0+} \frac{\vec{\Psi}(B_R(\vec{x}))}{M(B_R(\vec{x}))}$$

where

$$B_R(\vec{x}) = \{\vec{y} \in R^3 \mid |\vec{x}-\vec{y}| < R\}.$$

Then $\vec{\Psi}$ is called a scalar (vector) extensive physical property ψ is the corresponding intensive (specific) property (quantity).

Example : $\vec{\Psi}$. linear momentum $\vec{P}$ $\Rightarrow$ $\vec{\psi}$ .. velocity $\vec{V}$

NOTE : Let $\mathcal{V} \subset V$ (a so called "control volume")

Then $\Psi(\mathcal{V}) = \int_{\mathcal{V}} \psi(\vec{x})\,dM = \int_{\mathcal{V}} \psi(\vec{x})\varrho(\vec{x})\,d\vec{x}$

ϱ ... material density

$\left(\psi = \frac{d\Psi}{dM} \right.$ Radon - Nikodym derivative $\left. \right)$

where $\varrho(\vec{x}) = \lim_{R \to 0+} \frac{M(B_R(\vec{x}))}{\underbrace{m_3(B_R(\vec{x}))}_{\frac{4}{3}\pi R^3}}$

notation : $\varrho(t, \vec{x}) = \rho(t, \vec{X})$ where $\vec{x} = \vec{x}(t, \vec{X})$

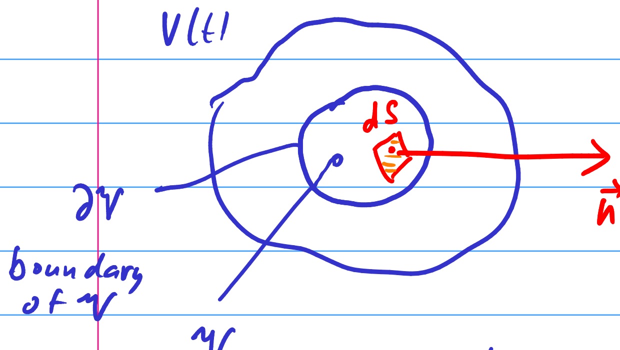
$$\Psi(\mathcal{V}_0) = \int_{\mathcal{V}_0} \psi(\mathbf{x})\,\mathrm{d}M = \int_{\mathcal{V}_0} \psi(\mathbf{x})\,\varrho(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

$$\boldsymbol{\Psi}(\mathcal{V}_0) = \int_{\mathcal{V}_0} \boldsymbol{\psi}(\mathbf{x})\,\mathrm{d}M = \int_{\mathcal{V}_0} \boldsymbol{\psi}(\mathbf{x})\,\varrho(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

CONSERVATION OF MASS ("COM")

- $\vec{X} \in V_0$, $\mathcal{V}_0 \subset V_0$ ... reference configuration of the material body (Lagrangian description)
- $\vec{x} \in V_0$, $\mathcal{V} \subset V(t)$ ... Eulerian description ... often convenient for study of fluids

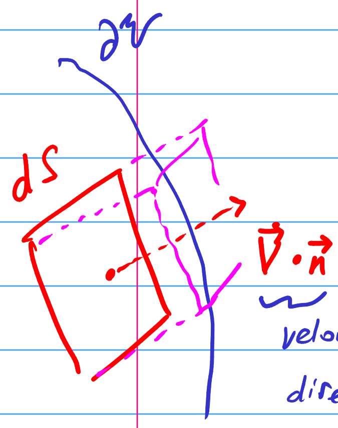
Let $\mathcal{V} \subset \mathbb{R}^3$, $(\mathcal{V} \subset V(t))$ be an arbitrarily chosen control volume



outer normal vector to the surface element $dS \subset \partial\mathcal{V}$

- mass contained inside $\mathcal{V}$ at time t

$$M(\mathcal{V}) = \int_{\mathcal{V}} \varrho(t,\vec{x})\,d\vec{x}$$



$$\bullet \; -\frac{dM}{dt}(\mathcal{V}) = \int_{\partial\mathcal{V}} \varrho(t,\vec{x})\,\vec{V}(t,\vec{x})\cdot\underbrace{\vec{n}\,dS}_{d\vec{S}} = $$

COM in integral form

$$= \int_{\mathcal{V}} \operatorname{div}(\varrho\vec{V})\,d\vec{x}$$

"decrease of mass in $\mathcal{V}$ per unit time = flux of mass across $\partial\mathcal{V}$ out from $\mathcal{V}$"

$$\Downarrow$$

$$-\frac{d}{dt}\int_{\mathcal{V}} \varrho(t,\vec{x})\,d\vec{x} = \int_{\mathcal{V}} \operatorname{div}(\varrho\vec{V})\,d\vec{x}$$

$$\int_{\mathcal{V}} \frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\vec{V})\,d\vec{x} = 0$$

$\Downarrow$ but $\mathcal{V}$ is arbitrary

$$\boxed{\left.\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\vec{V}) = 0\right|_{(t,\vec{x})}}$$

continuity equation (COM in differential form)

Reynolds Transport Theorem

Lemma: $$\left.\frac{\partial(|\det \mathbb{F}|)}{\partial t}\right|_{(t,\vec{X})} = |\det \mathbb{F}|\Big|_{(t,\vec{X})} \left.\nabla\cdot\vec{V}\right|_{(t,\vec{x}(t,\vec{X}))}$$

$$\mathbb{F} = \left.\left(\frac{\partial x_i}{\partial X_j}\right)\right|_{(t,\vec{X})}$$

deformation gradient

Proof: $$\left.\frac{\partial(\det \mathbb{F})}{\partial t}\right|_{(t,\vec{X})} = \left.\frac{\partial}{\partial t}\left(\frac{1}{3!}\varepsilon_{IJK}\,\varepsilon_{ijk}\,\underbrace{\frac{\partial x_I}{\partial X_i}}_{\mathbb{F}_{Ii}}\cdot\underbrace{\frac{\partial x_J}{\partial X_j}}_{\mathbb{F}_{Jj}}\cdot\underbrace{\frac{\partial x_K}{\partial X_k}}_{\mathbb{F}_{Kk}}\right)\right|_{(t,\vec{X})} =$$

$$= \left.\frac{1}{3!}\varepsilon_{IJK}\,\varepsilon_{ijk}\left(\frac{\partial v_I}{\partial X_i}\frac{\partial x_J}{\partial X_j}\frac{\partial x_K}{\partial X_k} + \underbrace{\frac{\partial x_I}{\partial X_i}\frac{\partial v_J}{\partial X_j}\frac{\partial x_K}{\partial X_k}}_{\substack{J\leftrightarrow I\\ j\leftrightarrow i}} + \underbrace{\frac{\partial x_I}{\partial X_i}\frac{\partial x_J}{\partial X_j}\frac{\partial v_K}{\partial X_k}}_{\substack{K\leftrightarrow I\\ k\leftrightarrow i}}\right)\right|_{(t,\vec{X})}$$

$$= \left.\frac{1}{2!}\varepsilon_{IJK}\,\varepsilon_{ijk}\frac{\partial v_I}{\partial X_i}\frac{\partial x_J}{\partial X_j}\frac{\partial x_K}{\partial X_k} = \frac{\partial v_I}{\partial X_i}\underbrace{\frac{1}{2!}\varepsilon_{IJK}\,\varepsilon_{ijk}\frac{\partial x_J}{\partial X_j}\frac{\partial x_K}{\partial X_k}}\right|_{(t,\vec{X})}$$

$$\boxed{\Delta_{Ii} = \tfrac{1}{2!}\varepsilon_{IJK}\,\varepsilon_{ijk}\,\alpha_{Jj}\,\alpha_{Kk}}$$

$$\Delta_{Ii} = \det\mathbb{F}\cdot\underbrace{\frac{\partial X_i}{\partial x_I}}_{\mathbb{F}^{-1}_{iI}}$$

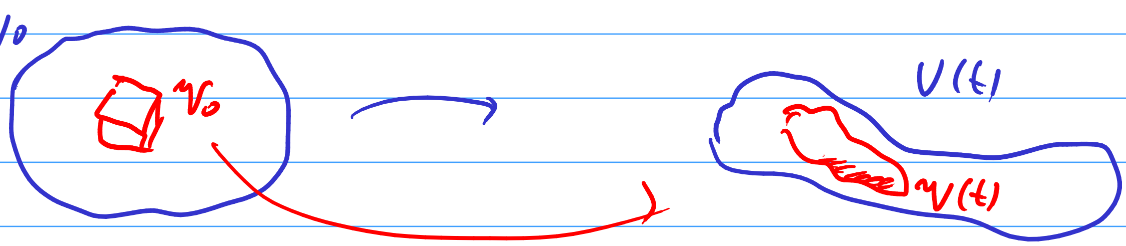
$$\mathbb{F}^{-1} = \left(\frac{\partial X_i}{\partial x_j}\right)$$

$$= \left.\frac{\partial V_I(t,\vec{x}(t,\vec{X}))}{\partial x_\ell}\left(\underbrace{\frac{\partial x_\ell}{\partial X_i}\frac{\partial X_i}{\partial x_I}}_{(\mathbb{F}\cdot\mathbb{F}^{-1})_{I\ell}=\delta_{I\ell}}\det\mathbb{F}\right)\right|_{(t,\vec{X})}$$

$$= \frac{\partial V_I(t,\vec{x}(t,\vec{X}))}{\partial x_I}\det\mathbb{F}\Big|_{(t,\vec{X})} = \det\mathbb{F}\Big|_{(t,\vec{X})}\;\nabla\cdot\vec{V}\Big|_{(t,\vec{x})}$$

<u>Reynolds T.T</u> : Let $\mathcal{V}_0 \subset V_0$ and some intensive physical property be described by $\phi : (0,T)\times V_0 \to \mathbb{R}$ $\wedge$ $\Phi : (0,T) \times V(t) \to \mathbb{R}$ satisfying $\Phi(t,\vec{x}) = \phi(t,\vec{X})$ where $\vec{x} = \vec{x}(t,\vec{X})$

denote $\quad V(t) = \vec{x}(t, V_0) \qquad \mathcal{V}(t) = \vec{x}(t, \mathcal{V}_0)$



then:

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}(t)} \Phi(t,\vec{x})\,d\vec{x}} = \frac{d}{dt}\int_{\mathcal{V}_0} \underbrace{\Phi(t,\vec{x}(t,\vec{X}))}_{\phi(t,\vec{X})} \,|\det \mathbb{F}|\, d\vec{X} =$$

$\hookrightarrow \vec{x} = \vec{x}(t,\vec{X})$

$$\mathbb{F} = \left(\frac{\partial x_i}{\partial X_j}\right)$$

$$= \frac{d}{dt}\int_{\mathcal{V}_0} \phi |\det \mathbb{F}|\, d\vec{X} = \int_{\mathcal{V}_0} \frac{\partial}{\partial t}\left(\phi |\det \mathbb{F}|\right) d\vec{X} = \int_{\mathcal{V}_0}\left[\frac{\partial \phi}{\partial t} + \phi\, \nabla\cdot\vec{V}(t,\vec{x}(t,\vec{X}))\right] |\det \mathbb{F}|\, d\vec{X}$$

$$\frac{\partial \phi}{\partial t}(t,\vec{X}) = \frac{D\Phi}{Dt}(t,\vec{x}(t,\vec{X}))$$

$$= \int_{\mathcal{V}(t)} \left(\frac{D\Phi}{Dt} + \Phi\, \nabla\cdot\vec{V}\right)\Bigg|_{(t,\vec{x})} d\vec{x} =$$

<u>Lemma:</u> $\dfrac{\partial}{\partial t}\left(\det \mathbb{F}\right) = \nabla\cdot\vec{V}\Big|_{(t,\vec{x})} \det \mathbb{F}\Big|_{(t,\vec{X})}$

$$= \int_{\mathcal{V}(t)} \underbrace{\frac{\partial \Phi}{\partial t} + \vec{V}\cdot\nabla\Phi + \Phi\,\nabla\cdot\vec{V}}_{\nabla\cdot(\Phi\vec{V})}\, d\vec{x} = \boxed{\int_{\mathcal{V}(t)} \frac{\partial \Phi}{\partial t} + \nabla\cdot\left(\Phi\vec{V}\right) d\vec{x}}$$

NOTE : Continuity equation derived using RTT

$$\Phi = \rho \Rightarrow \underbrace{\frac{d}{dt}\int_{V(t)} \rho(t,\vec{x})\,d\vec{x}}\ = 0 = \int_{V(t)} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\,d\vec{x}$$

COM formulation : $M(V(t))$ = mass inside $V(t)$

= mass inside V_0 $= M(V_0) = \int_{V_0} \rho(t,\vec{X})\,d\vec{X}$

$\Leftrightarrow$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V} = \underline{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0}$$

... again, we get the continuity eq. in spatial coordinates

RTT for specific quantities :

define $\Phi = \rho F$ where F is an arbitrary function & ρ is density

$$\Rightarrow \frac{d}{dt}\int_{V(t)} \rho F\,d\vec{x} = \dots = \int_{V(t)} \frac{D(\rho F)}{Dt} + (\rho F)\nabla\cdot\vec{V}\,d\vec{x} =$$

$$= \int_{V(t)} \frac{D\rho}{Dt}F + \rho\frac{DF}{Dt} + \rho F\nabla\cdot\vec{V}\,d\vec{x} = \int_{V(t)} \rho\frac{DF}{Dt}\,dx$$

$$F\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V}\right) = 0$$

STREAMLINES & TRAJECTORIES

- **trajectory** of a material point $\vec{X} \in V_0$ is a curve

$$\varphi = \{ \vec{x}(t, \vec{X}) \mid t \in \langle 0, T \rangle \}$$

with a "natural" parametrization $\vec{\varphi}(t) = \vec{x}(t, \vec{X}) \quad \Big| \frac{\partial}{\partial t}$ for $\vec{X}$ fixed

$$\boxed{\begin{aligned} &\frac{d\vec{\varphi}}{dt} = \vec{v}(t, \vec{X}) = \vec{V}(t, \vec{\varphi}(t)) \\ &\vec{\varphi}(0) = \vec{X} \end{aligned}}$$

- **streamline** is a curve tangent to the velocity field $\vec{V}(t, \vec{x})$ at a given fixed time t and in every point $\vec{x}$

A parametrization of a streamline $\tilde{\varphi}$ crossing the point $\vec{x} \in V(t)$

is given by the map $\vec{\tilde{\varphi}}$ satisfying

WLOG: $\alpha = 1$ $\quad \tilde{s} = h(s)$

$\vec{\tilde{\varphi}} : s \mapsto V(t)$ a

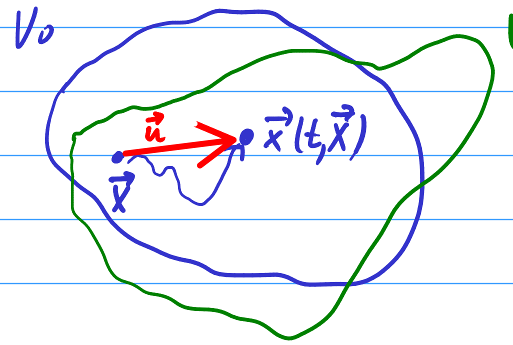
$$\boxed{\begin{aligned} &\frac{d\vec{\tilde{\varphi}}(s)}{ds} =_{\alpha} \vec{V}(t, \vec{\tilde{\varphi}}(s)) \\ &\vec{\tilde{\varphi}}(0) = \vec{x} \end{aligned}}$$

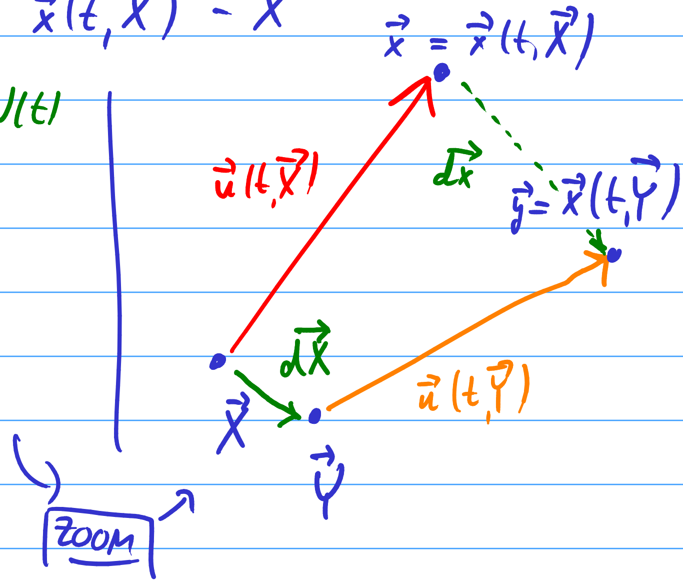
$\alpha > 0$

If $\vec{V}$ is independent of time, streamlines & trajectories coincide (we speak of "stationary" flow)

DEFORMATIONS IN FLUIDS

- displacement vector $\vec{u}(t,\vec{X}) = \vec{x}(t,\vec{X}) - \vec{X}$





for $\vec{X},\vec{Y} \in V_0$, we define

$$d\vec{X} = \vec{Y} - \vec{X}$$
$$\vec{X} + d\vec{X} = \vec{Y}$$

$$\vec{x} = \vec{x}(t,\vec{X}) = \vec{X} + \vec{u}(t,\vec{X})$$
$$\vec{y} = \vec{x}(t,\vec{Y}) = \vec{Y} + \vec{u}(t,\vec{Y})$$

$$\begin{aligned} d\vec{x} = \vec{y} - \vec{x} &= \vec{x}(t,\vec{X}+d\vec{X}) - \vec{x}(t,\vec{X}) = \\ &= \vec{x}(t,\vec{X}) + \nabla\vec{x}(t,\vec{X})\cdot d\vec{X} + \vec{\alpha}(d\vec{X}) - \vec{x}(t,\vec{X}) \\ &= \nabla\vec{x}(t,\vec{X})\cdot d\vec{X} + \vec{\alpha}(d\vec{X}) \\ &= \mathbb{F}(t,\vec{X})\cdot d\vec{X} + \vec{\alpha}(d\vec{X}) \end{aligned}$$

$$\nabla\vec{x} = (\partial_i x_j) \leftarrow \mathbb{F} = \left(\frac{\partial x_i}{\partial X_j}\right)$$

$$\lim_{\vec{h}\to 0} \frac{\vec{\alpha}(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

Component-wise: $dx_i = \frac{\partial x_i}{\partial X_j} dX_j + o(\|d\vec{X}\|)$

def. $\boxed{\mathbb{H} = \nabla u = \left(\frac{\partial u_i}{\partial X_j}\right) = \left(\frac{\partial (x_i - X_i)}{\partial X_j}\right) = \left(\frac{\partial x_i}{\partial X_j} - \delta_{ij}\right) = \mathbb{F} - \mathbb{I}}$

then, substituting to $d\vec{x} = \mathbb{F} \cdot d\vec{X} + \vec{o}(d\vec{X})$,

we get:

$$d\vec{x} = (\mathbb{H} + \mathbb{I}) d\vec{X} + \vec{o}(d\vec{X}) = d\vec{X} + \nabla\vec{u} \cdot d\vec{X} + \vec{o}(d\vec{X})$$

Lagrangian strain tensor

$$\|d\vec{x}\|^2 = \|d\vec{X} + \nabla\vec{u} \cdot d\vec{X} + \vec{o}(d\vec{X})\|^2 =$$

$$\left(dX_i + \frac{\partial u_i}{\partial X_j} dX_j + o(d\vec{X})\right)\left(dX_i + \frac{\partial u_i}{\partial X_k} dX_k + o(d\vec{X})\right) =$$

$$= dX_i dX_i + \frac{\partial u_i}{\partial X_j} dX_j dX_i + \underbrace{\frac{\partial u_i}{\partial X_k} dX_i dX_k}_{i \to j,\ k \to i} + \underbrace{\frac{\partial u_i}{\partial X_j} \frac{\partial u_i}{\partial X_k} dX_j dX_k}_{i \leftrightarrow k} + o(d\vec{X})$$

$$= \|d\vec{X}\|^2 + \frac{\partial u_i}{\partial X_j} dX_j dX_i + \frac{\partial u_j}{\partial X_i} dX_j dX_i + \frac{\partial u_k}{\partial X_j} \frac{\partial u_k}{\partial X_i} dX_j dX_i + o(d\vec{X})$$

$$= \|d\vec{X}\|^2 + 2\varepsilon_{ij}\, dX_j dX_i + o(d\vec{X})$$, where

$$\mathbb{e}(t, \vec{X}) = (\varepsilon_{ij}) = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \underbrace{\frac{\partial u_k}{\partial X_j} \frac{\partial u_k}{\partial X_i}}_{\text{can be neglected}}\right)\right)$$

Lagrangian strain tensor

$\downarrow$

$$\tilde{\mathbb{e}}(t, \vec{X}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

... infinitesimal strain tensor (tensor of small deformations)

$$\nabla\vec{u} = \mathbb{H} = \left(\frac{\partial u_i}{\partial X_j}\right) = \underbrace{\overbrace{\tfrac{1}{2}\left(\mathbb{H}+\mathbb{H}^T\right)}^{\nabla\vec{u}_{sym}}}_{\substack{\text{sym. tens.} \\ \tilde{e}}} + \underbrace{\overbrace{\tfrac{1}{2}\left(\mathbb{H}-\mathbb{H}^T\right)}^{\nabla\vec{u}_{skew}}}$$

$$d\vec{x} - o(d\vec{X}) = d\vec{X} + \nabla\vec{u}\cdot d\vec{X} = d\vec{X} + \nabla\vec{u}_{sym}\cdot d\vec{X} + \nabla\vec{u}_{skew}\cdot d\vec{X}$$

$$= d\vec{X} + \tilde{e}\cdot d\vec{X} + \underline{\vec{w}\times d\vec{X}}$$

where $\nabla\vec{u}_{skew} = \begin{pmatrix} 0 & -w_3 & -w_2 \\ w_3 & 0 & -w_1 \\ w_2 & w_1 & 0 \end{pmatrix}$

$\vec{w} = (w_i) = \nabla\times\vec{u}$ $(= \text{rot}\,\vec{u})$

$\vec{x} = \vec{x}(t,\vec{X}) = \vec{X} + \vec{u}(t,\vec{X})$

$\vec{y} = \vec{x}(t,\vec{Y}) = \vec{Y} + \vec{u}(t,\vec{Y})$

$$\underbrace{\vec{u}(t,\vec{Y})}_{\vec{y}-\vec{Y}} = \underbrace{\vec{u}(t,\vec{X})}_{\vec{x}-\vec{X}} + \underbrace{\vec{y}-\vec{x}}_{d\vec{x}} - \underbrace{(\vec{Y}-\vec{X})}_{d\vec{X}} = \vec{u}(t,\vec{X}) + d\vec{x} - d\vec{X} =$$

$$= \vec{u}(t,\vec{X}) + \tilde{e}\cdot d\vec{X} + \vec{w}\times d\vec{X} \quad \text{here } \vec{w} = \text{rot}\,\vec{u}(t,\vec{X})$$

$$+\, o(d\vec{X})$$

decomposition of displacement of $\vec{Y}$ w.r.t. $\vec{X}$

- $\vec{u}(t,\vec{X})$: displacement of $\vec{X}$
- $\tilde{e}\cdot d\vec{X}$: stretching in the direction of eigenvectors of $\tilde{e}$ ($\tilde{e}$ is symmetric and thus diagonalizable, with real spectrum)
- $\vec{w}\times d\vec{X}$: rotation of $\vec{Y}$ about the point $\vec{X}$
- $o(d\vec{X})$: higher order deformations

STRAIN RATE TENSOR

$\nabla\vec{u} = \left(\frac{\partial u_i}{\partial X_j}\right)$ $\quad \frac{d}{dt}(\nabla\vec{u}) = \left(\frac{\partial \dot{u}_i}{\partial X_j}\right) = \left(\frac{\partial v_i}{\partial X_j}\right)$ $\quad \vec{u} = \vec{x}(t,\vec{X}) - \vec{X}$

$$\frac{d}{dt}(d\vec{x}) = \frac{d}{dt}\left(\nabla\vec{u}(t,\vec{X})\cdot d\vec{X}\right) = \nabla\vec{v}(t,\vec{X})\cdot d\vec{X} = \left(\frac{\partial v_i}{\partial X_j}\,dX_j\right)$$

$+\, o(d\vec{X})$ $\quad + \, o(d\vec{X})$

$\nabla\vec{v}$: velocity gradient in material coordinates

$d\vec{x} = d\vec{X} + \nabla\vec{u}\cdot d\vec{X}$

$$\frac{d(d\vec{x})}{dt} = \left(\frac{\partial v_i}{\partial X_j} dX_j\right) = \Bigg(\frac{\partial V_i}{\partial x_k} \overbrace{\frac{\partial x_k}{\partial X_j} dX_j}^{dx_k \, + o(d\vec{X})}\Bigg) = \underbrace{\nabla \vec{V}(t,\vec{x})}_{\text{velocity gradient in spatial coordinates}} \cdot d\vec{x} + o(d\vec{x})$$

$$\frac{d}{dt}\|d\vec{x}\|^2 = 2\,\|d\vec{x}\|\,\frac{d}{dt}\|d\vec{x}\|$$

rate of relative deformation (strain)
$$\frac{\frac{d}{dt}\|d\vec{x}\|}{\|d\vec{x}\|} = \frac{1}{2}\,\frac{\frac{d}{dt}\|d\vec{x}\|^2}{\|d\vec{x}\|^2} = \frac{1}{2}\,\frac{2\,\dot{\varepsilon}_{ij}\, dX_i dX_j}{\|d\vec{x}\|^2}$$

let's manipulate the numerator:

$$2\,\dot{\varepsilon}_{ij}\, dX_i dX_j = \frac{d}{dt}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) dX_i dX_j =$$

$$= \left(\frac{\partial v_i}{\partial X_j} + \underbrace{\frac{\partial v_j}{\partial X_i}}_{i\leftrightarrow j} + \frac{\partial v_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \underbrace{\frac{\partial u_k}{\partial X_i}\frac{\partial v_k}{\partial X_j}}_{i\leftrightarrow j}\right) dX_i dX_j =$$

$$= 2\left(\frac{\partial v_j}{\partial X_i} + \frac{\partial v_k}{\partial X_j}\frac{\partial u_k}{\partial X_i}\right) dX_i dX_j =$$

we use $\mathbb{H} = \mathbb{F} - \mathbb{I}$

$u_i = x_i - X_i$

i.e. $\dfrac{\partial u_i}{\partial X_j} = \dfrac{\partial x_i}{\partial X_j} - \delta_{ij}$

$\vec{u} = \vec{x} - \vec{X}$ $\qquad \dfrac{\partial X_k}{\partial X_i} = \delta_{ik}$

$$= 2\left(\cancel{\frac{\partial v_j}{\partial X_i}} + \underbrace{\frac{\partial v_k}{\partial X_j}}\left(\frac{\partial x_k}{\partial X_i} \cancel{-\delta_{ik}}\right)\right) dX_i dX_j =$$

$$= 2\,\frac{\partial v_k}{\partial X_j}\frac{\partial x_k}{\partial X_i}\, dX_i dX_j = 2\,\frac{\partial V_k}{\partial x_\ell}\frac{\partial x_\ell}{\partial X_j}\overbrace{\frac{\partial x_k}{\partial X_i} dX_i}^{dx_k + o(d\vec{x})} dX_j =$$

$$\frac{\partial x_\ell}{\partial X_j} dX_j = dx_\ell + o(d\vec{x})$$

$$= 2\,\frac{\partial V_k}{\partial x_\ell}\, dx_k dx_\ell + o\left(\|d\vec{x}\|^2\right)$$

$$= \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) dx_i dx_j + o\left(\|d\vec{x}\|^2\right)$$

$$= \mathbb{D}\vec{x}$$

$$\Rightarrow \frac{\frac{d}{dt}(\|d\vec{x}\|)}{\|d\vec{x}\|} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{d_{ij}} \underbrace{\frac{dx_i\, dx_j}{dx_k\, dx_k}}_{\alpha_i \alpha_j} + \underbrace{\dots}_{\to 0 \text{ for } \|d\vec{x}\| \to 0}$$

where $\vec{\alpha} = \frac{d\vec{x}}{\|d\vec{x}\|}$

$(\nabla \vec{V})_{sym} = \mathbb{D} = (d_{ij})$ ... strain rate tensor

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FLUID DYNAMICS – FORCES IN CONTINUUM

- Cauchy: there are volume & surface forces in continuum:
  - specific volume forces vs. force density

$$\vec{F}(\vec{x}) = \lim_{R\to 0+} \frac{\vec{F}_V(B_R(\vec{x}))}{M(B_R(\vec{x}))} =$$

force per unit mass

volume force acting on the ball $B_R(\vec{x})$ (... vector measure)

mass of the ball $B_R(\vec{x})$

$$= \lim_{R\to 0+} \frac{\vec{F}_V(B_R(\vec{x}))}{\frac{4}{3}\pi R^3} \cdot \underbrace{\frac{\frac{4}{3}\pi R^3}{M(B_R(\vec{x}))}}_{\to \frac{1}{\varrho(\vec{x})}} = \frac{1}{\varrho(\vec{x})} \lim_{R\to 0+} \underbrace{\frac{\vec{F}_V(B_R(\vec{x}))}{\frac{4}{3}\pi R^3}}_{\text{force per unit volume (force density)}}$$

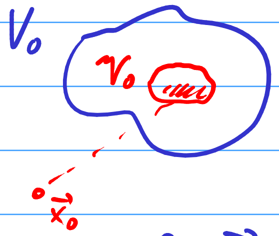
  - Cauchy's postulate for surface forces: surface force per unit surface area acting on the surface element dS with normal $\vec{n}$ is

$$\vec{T}(t, \vec{x}, \vec{n}(\vec{x}))\, dS$$

... i.e. $\vec{T}$ does not depend e.g. on curvature ($\nabla \cdot \vec{n}$)

Note: In 1957, Cauchy's postulate was proved by Walter Noll

LAW OF FORCE (NEWTON'S 2nd LAW) IN CONTINUUM



again, choose the control volume $\mathcal{V}_0$ in the material body V_0, which evolves as $\mathcal{V}(t) = \vec{x}(t, \mathcal{V}_0)$

specific volume force

law of force (Impulse-momentum theorem)

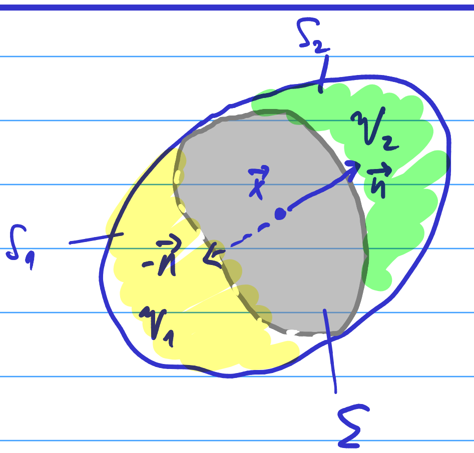
$$\frac{d}{dt}\int_{\mathcal{V}(t)} \varrho \vec{V}\, d\vec{x} = \int_{\mathcal{V}(t)} \varrho \vec{F}\, d\vec{x} + \int_{\partial\mathcal{V}(t)} \vec{T}(t,\vec{x},\vec{n}(\vec{x}))\, dS$$

or: $(\vec{x}-\vec{x}_0)$ for $\vec{x}_0$ fixed (but arbitrary)

balance of angular momentum w.r.t. $\vec{x}_0$ (or $\vec{0}$)

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \varrho \vec{x}\times\vec{V}\, d\vec{x} = \int_{\mathcal{V}(t)} \varrho\, \vec{x}\times\vec{F}\, d\vec{x} + \int_{\partial\mathcal{V}(t)} \vec{x}\times\vec{T}(t,\vec{x},\vec{n}(\vec{x}))\, dS$$

Cauchy's (fundamental) lemma



$\mathcal{V}_0 = \mathcal{V}_1 \cup \mathcal{V}_2$

$\partial\mathcal{V}_0 = S_1 \cup S_2$

$\vec{x} \in \Sigma$

$\vec{n}$ ... unit normal vector to the surface Σ pointing outward w.r.t. $\mathcal{V}_1$

$-\vec{n}$ .... — " — w.r.t. $\mathcal{V}_2$

momentum balances for $\mathcal{V}_1$ and $\mathcal{V}_2$:

$$\mathcal{V}_1 \qquad \frac{d}{dt}\int_{\vec{x}(t,\mathcal{V}_1)} \varrho \vec{V}\, d\vec{x} = \int_{\vec{x}(t,\mathcal{V}_1)} \varrho \vec{F}\, d\vec{x} + \int_{\vec{x}(t,\Sigma)} \vec{T}(t,\vec{x},\vec{n}(\vec{x}))\, dS + \int_{\vec{x}(t,S_1)} \vec{T}(t,\vec{x},\vec{n}(\vec{x}))\, dS$$

$\oplus$

$\mathcal{V}_2$ $$\frac{d}{dt}\int_{\vec{x}(t,\mathcal{V}_2)} \rho \vec{V} d\vec{x} = \int_{\vec{x}(t,\mathcal{V}_2)} \rho \vec{F} d\vec{x} + \int_{\vec{x}(t,\Sigma)} \vec{T}(t,\vec{x},-\vec{n}(\vec{x}))dS + \int_{\vec{x}(t,S_2)} \vec{T}(t,\vec{x},\vec{n}(\vec{x}))dS$$

$\ominus$

$\mathcal{V}$ $$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho \vec{V} d\vec{x} = \int_{\mathcal{V}(t)} \rho \vec{F} d\vec{x} + \int_{\partial\mathcal{V}(t)} \vec{T}(t,\vec{x},\vec{n}(\vec{x}))dS$$

$$0 = \int_{\vec{x}(t,\Sigma)} \left[\vec{T}(t,\vec{x},\vec{n}(\vec{x})) + \vec{T}(t,\vec{x},-\vec{n}(\vec{x}))\right] dS$$

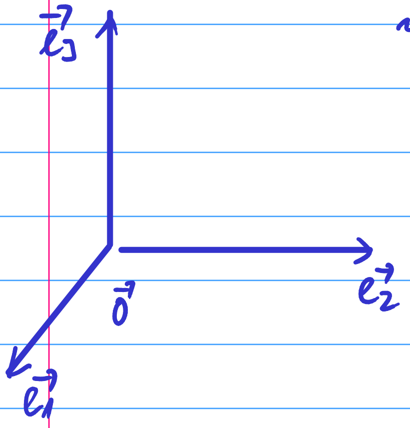
$\Downarrow$ for any $\mathcal{V}_0$, Σ (for $\vec{x}\in\Sigma$, Σ can be chosen such that $\vec{n}(\vec{x})$ points in an arbitrary direction)

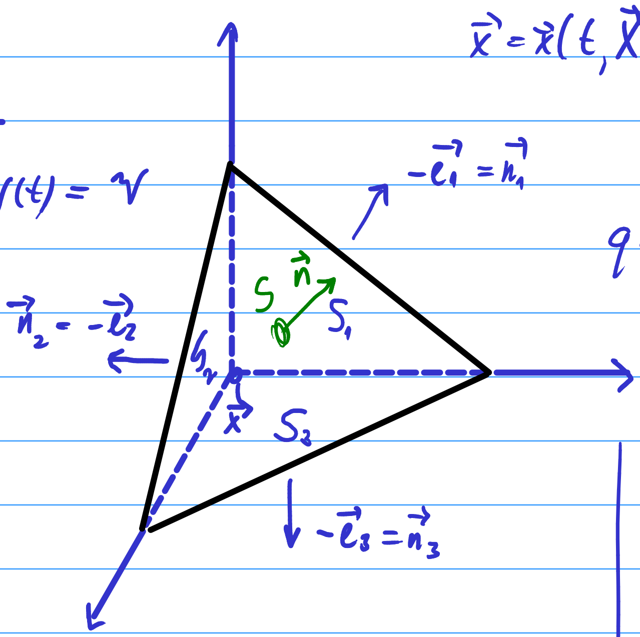
$$\boxed{\vec{T}(t,\vec{x},\vec{n}) = -\vec{T}(t,\vec{x},-\vec{n})}$$

... law of action & reaction in fluids

$\vec{x} = \vec{x}(t, \vec{X}) \quad \vec{X} \in V_0$

STRESS TENSOR





quadrilateral

$\|\vec{n}\| = 1$

$\vec{n} \cdot \vec{e}_i = n_i$

$\vec{n} = (n_i)$

surface forces acting on $\partial\mathcal{V}$:

$$\int_{\partial\mathcal{V}} T\,dS = \vec{T}(t, \vec{\xi}_1, \vec{n}_1) \cdot |S_1| + \vec{T}(t, \vec{\xi}_2, \vec{n}_2) \cdot |S_2| + \vec{T}(t, \vec{\xi}_3, \vec{n}_3) \cdot |S_3|$$

$$+ \vec{T}(t, \vec{\xi}, \vec{n})\,|S|$$

where $\vec{\xi}_i \in S_i$, $\vec{n}_i = -\vec{e}_i$ (mean value theorem) ($\vec{T}$ is continuous)

$$= |S| \left(-\vec{T}(t, \vec{\xi}_1, \vec{e}_1) \underbrace{\frac{|S_1|}{|S|}}_{n_1} - \vec{T}(t, \vec{\xi}_2, \vec{e}_2)\, n_2 - \vec{T}(t, \vec{\xi}_3, \vec{e}_3)\, n_3 + \vec{T}(t, \vec{\xi}, \vec{n}) \right)$$

momentum balance in $\mathcal{V}$:

where $\tilde{\vec{\xi}}, \hat{\vec{\xi}} \in \mathcal{V}$

$$\frac{d}{dt}\left((\rho \vec{V})\Big|_{(t, \tilde{\vec{\xi}})} |\mathcal{V}| \right) = \rho \vec{F}\Big|_{(t, \hat{\vec{\xi}})} |\mathcal{V}| - \left(\sum_i \vec{T}(t, \vec{\xi}_i, \vec{e}_i)\, n_i + \vec{T}(t, \vec{\xi}, \vec{n}) \right) |S|$$

Now, let's scale $\mathcal{V}$ w.r.t. the point $\vec{x}$ by a factor ε

$|S| = \mathcal{O}(\varepsilon^2)$, $|\mathcal{V}| = \mathcal{O}(\varepsilon^3) = o(\varepsilon^2)$

divide the equality by ε^2 and pass to the limit $\varepsilon \to 0$:

$$\Rightarrow \boxed{\sum_i \vec{T}(t, \vec{x}, \vec{e}_i)\, n_i + \vec{T}(t, \vec{x}, \vec{n}) = \vec{0}}$$

$$\Rightarrow \quad \vec{T}(t,\vec{x},\vec{n}) = \underbrace{\left(\vec{T}(t,\vec{x},\vec{e}_1)\ \vec{T}(t,\vec{x},\vec{e}_2)\ \vec{T}(t,\vec{x},\vec{e}_3) \right)}_{\mathbb{T}(t,\vec{x}) = (\tau_{ij})} \vec{n}$$

$\Rightarrow$ there exists the stress tensor

SYMMETRY OF THE STRESS TENSOR

in general: $(\vec{x}-\vec{x}_0)$ for any $\vec{x}_0$ (fixed)

angular momentum balance

$$\frac{d}{dt}\int_{V(t)} \vec{x} \times \varrho \vec{V}\, d\vec{x} = \int_{V(t)} \varrho\, \vec{x} \times \vec{F}\, d\vec{x} + \int_{\partial V(t)} \vec{x} \times \vec{T}(t,\vec{x},\vec{n}(\vec{x}))\, dS$$

WLOG w.r.t. $\vec{0}$

def. $(*)$

$$\int_{\partial V(t)} \vec{x} \times \vec{T}(t,\vec{x},\vec{n}(\vec{x}))\, dS = \int_{\partial V(t)} \vec{x} \times \left(\mathbb{T}(t,\vec{x}) \cdot \vec{n}\right) dS = \int_{\partial V(t)} \varepsilon_{ijk}\, x_j\, (\mathbb{T}\vec{n})_k\, \vec{e}_i\, dS$$

$$= \varepsilon_{ijk}\, \vec{e}_i \int_{\partial V(t)} x_j\, \tau_{k\ell}\, n_\ell\, dS = \varepsilon_{ijk}\, \vec{e}_i \int_{V(t)} \partial_\ell (x_j \tau_{k\ell})\, d\vec{x}$$

Green's formula

$$\boxed{\int_V \underbrace{(\partial_i f)}_{\frac{\partial f}{\partial x_i}} g\, dx = -\int_V f\, \partial_i g\, dx + \int_{\partial V} f g\, n_i\, dS}$$

$g(\vec{x}) = 1$ this vanishes

$$= \varepsilon_{ijk}\, \vec{e}_i \int_{V(t)} \big(\underbrace{\delta_{j\ell}\, \tau_{k\ell} + x_j\, \partial_\ell \tau_{k\ell}}_{(\partial_\ell x_j)\tau_{k\ell} + x_j \partial_\ell \tau_{k\ell}} \big)\, d\vec{x} = \varepsilon_{ijk}\, \vec{e}_i \int_{V(t)} \left(\tau_{kj} + x_j\, \partial_\ell \tau_{k\ell}\right) d\vec{x}$$

d.l.e:

by the Reynolds Transport Theorem

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \varrho \vec{x}\times\vec{V}\,d\vec{x} = \int_{\mathcal{V}(t)} \varrho \frac{D}{Dt}(\vec{x}\times\vec{V})\,d\vec{x} = \int_{\mathcal{V}(t)} \varrho \Big(\underbrace{\frac{D\vec{x}}{Dt}\times\vec{V}}_{\vec{V}\times\vec{V}=\vec{0}} + \vec{x}\times\frac{D\vec{V}}{Dt}\Big)d\vec{x}$$

$\vec{x}(t,\vec{X})$

$$= \int_{\mathcal{V}(t)} \varrho \vec{x}\times\frac{D\vec{V}}{Dt}\,d\vec{x}$$

$$\frac{Dx_i}{Dt} = \underbrace{\frac{\partial x_i}{\partial t}}_{0} + V_j \underbrace{\partial_j x_i}_{\delta_{ij}} = V_i$$

$$\frac{D\vec{W}}{Dt} = \frac{\partial\vec{w}}{\partial t} \qquad x_i = x_i(t,\vec{X})$$

now, plug this back to (*)

$$\underbrace{\int_{\mathcal{V}(t)} \varrho\vec{x}\times\frac{D\vec{V}}{Dt}\,d\vec{x}}_{\to 0} = \underbrace{\int_{\mathcal{V}(t)} \varrho\,\vec{x}\times\vec{F}\,d\vec{x}}_{\to 0} + \varepsilon_{ijk}\,\vec{e}_i \int_{\mathcal{V}(t)} \big(\tau_{kj} + \underbrace{x_j\partial_l\tau_{kl}}_{\to 0}\big)\,d\vec{x}$$

Now assume that $\vec{0}\in\mathcal{V}(t)$ and scale $\mathcal{V}$ by the factor ε with respect to $\vec{0}$ (or generally, w.r.t $\vec{x}_0$)

Then $\|\vec{x}\| = \mathcal{O}(\varepsilon)$. $|\mathcal{V}| = \mathcal{O}(\varepsilon^3)$. Divide by ε^3 and then $\varepsilon\to 0$

$$\Rightarrow \boxed{\varepsilon_{ijk}\,\vec{e}_i\,\tau_{kj} = \vec{0}} = \sum_{i=1}^{3}\varepsilon_{ijk}\tau_{kj}\vec{e}_i = \begin{pmatrix}\varepsilon_{1jk}\tau_{kj}\\ \varepsilon_{2jk}\tau_{kj}\\ \varepsilon_{3jk}\tau_{kj}\end{pmatrix}$$

1. component: $\varepsilon_{1jk}\tau_{kj} = 0 \iff \tau_{32} = \tau_{23}$

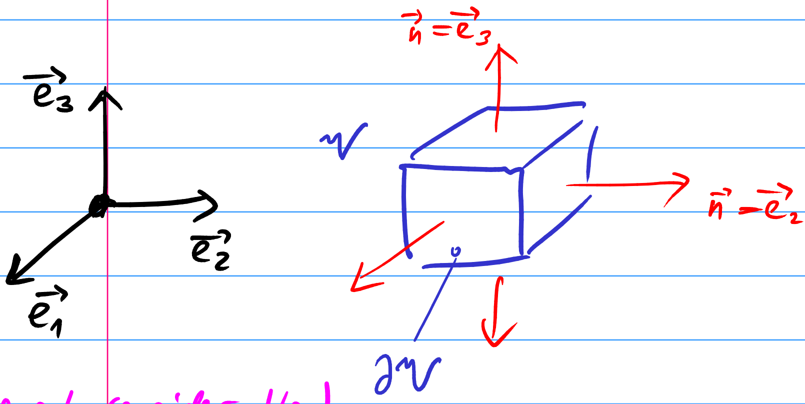
2. $\varepsilon_{2jk}\tau_{kj} = 0 \iff \tau_{31} = \tau_{13}$

3. — " — $= 0 \qquad \tau_{12} = \tau_{21}$

$\Rightarrow \underline{\underline{\mathbb{T} = \mathbb{T}^T}}$

$$\Leftrightarrow \underline{\underline{\mathbb{T} = \mathbb{T}^T}} \quad (\mathbb{T} \text{ is symmetric})$$

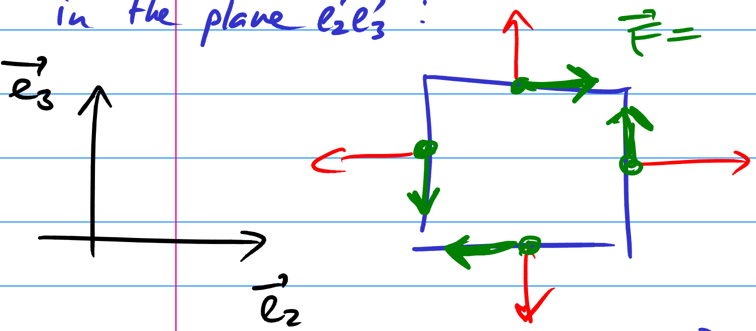
NOTE: Interpretation of the symmetry of $\mathbb{T}$



τ_{ij} is the i-th component of the force acting on (unit) surface with normal vector $\vec{e}_j$

(Let's only consider the) Tangent components in the plane $\vec{e}_2 \vec{e}_3$:

$\downarrow$ 2D view



$$\vec{F} = \begin{pmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ (e.g.)}$$

$$\vec{F} = \begin{pmatrix} \tau_{12} \\ \tau_{22} \\ \tau_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

e.g.

$$\mathbb{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\tau_{23} = \tau_{32}$$

$\Rightarrow$ infinitesimally small volume elements are not pushed to start rotating

small $\square$ $\Rightarrow$ $\mathbb{T}$ has almost the same value on the whole $\partial\mathcal{V}$

POZN: off-diagonal elements $\Rightarrow$ forces in the tangential direction (viscous forces)

diagonal elements $\Rightarrow$ forces in the normal direction (pressure)

MOMENTUM CONSERVATION LAW IN GENERAL FORM

we have:
$$\frac{d}{dt}\int_{V(t)} \rho \vec{V}\, d\vec{x} = \int_{\partial V(t)} \vec{T}(t,\vec{x},\vec{n})\, dS + \int_{V(t)} \rho \vec{F}\, d\vec{x} \Bigg|_{(t,\vec{x})}$$

componentwise (i-th component)

$$\frac{d}{dt}\int_{V(t)} \rho V_i\, d\vec{x} = \int_{\partial V(t)} T_i(t,\vec{x},\vec{n})\, dS + \int_{V(t)} \rho F_i\, d\vec{x}$$

use the RTT with the choice $\Phi = \rho V_i$

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho V_i) + \nabla\cdot(\rho V_i \vec{V})\, d\vec{x} = \int_{\partial V(t)} \tau_{ij} n_j\, dS + \int_{V(t)} \rho F_i\, d\vec{x}$$

$\downarrow$ Einstein's summation rule $\qquad \downarrow$ Green's formula

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho V_i) + \partial_j(\rho V_i V_j)\, d\vec{x} = \int_{V(t)} \partial_j \tau_{ij} + \rho F_i\, d\vec{x}$$

... all this for $\forall V_0 \iff \forall V(t)$

$\Downarrow$

"COMO"
$$\frac{\partial}{\partial t}(\rho V_i) + \partial_j(\rho \underbrace{V_i V_j}_{(\vec{V}\otimes\vec{V})_{ij}}) = \partial_j \tau_{ij} + \rho F_i \qquad i = 1,2,3$$

"COMO" in differential (vector) form

$$\boxed{\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla\cdot(\rho \vec{V}\otimes\vec{V}) = \nabla\cdot\mathbb{T} + \rho\vec{F}}$$

conservative form

Alternatively, for RTT in the form : $\frac{d}{dt}\int_{\mathcal{V}(t)} \varrho \phi \, d\vec{x} = \int_{\mathcal{V}(t)} \varrho \frac{D\phi}{Dt} \, d\vec{x}$

$$\boxed{\varrho \frac{D\vec{V}}{Dt} = \nabla\cdot\mathbb{T} + \varrho\vec{F}}$$

non-conservative form

SIMPLE FLUIDS

a fluid is called "SIMPLE" $\Leftrightarrow$ $\underbrace{\mathbb{T} = -P\mathbb{I} + \mathbb{T}_D}_{\tau_{ij} = -P\delta_{ij} + \tilde{\tau}_{ij}}$

where $\mathbb{T}_D = (\tilde{\tau}_{ij})$ is the dynamic (viscous) stress tensor

and $\mathbb{T}_D$ is only a function of fluid velocity and its derivatives. In addition, if $\vec{V}$ and its derivatives at the point $\vec{x}$ are zero, then $\mathbb{T}_D(t,\vec{x}) = \Theta$.

ve směru normály

P has the meaning of hydrostatic (thermodynamic) pressure exerted on the surface of any control volume in the normal direction. P is given by the equation of state (EOS)

$$P = P(\varrho, T)$$

$$-\partial_j \delta_{ij} P = -\partial_i P$$

"COMO" for a simple fluid

$$\frac{\partial}{\partial t}(\varrho V_i) + \partial_j(\varrho V_i V_j) = -\partial_i P + \partial_j \tilde{\tau}_{ij} + \varrho F_i$$

$$\boxed{\frac{\partial}{\partial t}(\varrho\vec{V}) + \nabla\cdot(\varrho\vec{V}\otimes\vec{V}) = -\nabla P + \nabla\cdot\mathbb{T}_D + \varrho\vec{F}}$$

Newtonian fluids & Navier–Stokes equations

Objective quantities : consider the transformation of the coordinate system (observer transformation)

$$\vec{x}'(t) = \mathbb{Q}(t)\vec{x} + \vec{C}(t)$$

$\mathbb{Q}$ ·· orthogonal

Def: a scalar quantity α is objective $\Leftrightarrow$ $\alpha(t,\vec{x}) = \alpha'(t,\vec{x}')$

a vector quantity $\vec{A}$ is objective $\Leftrightarrow$ $\|\vec{A}(t,\vec{x})\| = \|\vec{A}'(t,\vec{x}')\|$

Example $\vec{A}$ ... relative position of points $\vec{x}, \vec{y}$ $\quad \vec{A} = \vec{y} - \vec{x}$

$$\boxed{\vec{A}' = \vec{y}' - \vec{x}' = \mathbb{Q}(t)\vec{y} + \vec{C}(t) - \left(\mathbb{Q}(t)\vec{x} - \vec{C}(t)\right) = \underline{\mathbb{Q}(t)\vec{A}}}$$

let $\mathbb{M}$ be a tensor such that $\left(\mathbb{M}\vec{A}\right)' = \mathbb{M}'\underbrace{\vec{A}'}_{\mathbb{Q}(t)\vec{A}} = \mathbb{Q}(t)\mathbb{M}\vec{A}$

$$\underline{\mathbb{M}' = \mathbb{Q}^T\mathbb{M}\mathbb{Q}}$$

$\mathbb{M} = \mathbb{M}(t,\vec{x}) \longrightarrow$ $\mathbb{M}$ is an objective tensor quantity $\Leftrightarrow$

<u>NOTES</u> :
- vector of relative (mutual) position is objective
- velocity & acceleration are NOT objective
- DÚ the symmetric part of the velocity gradient $\mathbb{V} = (\partial_j V_i)$ is objective

  but that's the strain rate tensor $\mathbb{D} = \left(\frac{1}{2}\left(\partial_j V_i + \partial_i V_j\right)\right)$

$\Rightarrow$ in order for $\mathbb{T}_D$ to be an objective tensor, it must depend on $\vec{V}$ only through the strain rate tensor $\mathbb{D}$

without any internal structure

- Newtonian fluid = isotropic fluid = $\mathbb{T}_D$ is an isotropic tensor function of $\mathbb{D}$
  - the dependence of $\mathbb{T}_D$ on $\mathbb{D}$ is LINEAR

we expect THIS to hold:

$$\boxed{Q^T \mathbb{T}_D(\mathbb{D}) Q = \mathbb{T}_D(Q^T \mathbb{D} Q)}$$

see the mathematical fundamentals above

for example, all powers of $\mathbb{D}$ are isotropic functions of $\mathbb{D}$

$$Q^T \mathbb{D}^n Q = Q^T \underbrace{\mathbb{D}\mathbb{D}\cdots\mathbb{D}}_{n\times} Q = (\underbrace{Q^T \mathbb{D} Q}_{\mathbb{D}'})^n \qquad Q^T Q = \mathbb{I}$$

consider the (rather general) dependence $\mathbb{T}_D(\mathbb{D}) = \sum_{n=0}^{N<+\infty} \tilde{\alpha}_n \mathbb{D}^n$ ✓ def #

where $\tilde{\alpha}_n = \tilde{\alpha}_n({}_D I_1, {}_D I_2, {}_D I_3)$

by the Cayley-Hamilton theorem: $\ell(\mathbb{D}) = -\mathbb{D}^3 + {}_D I_1 \mathbb{D}^2 - {}_D I_2 \mathbb{D} + {}_D I_3 \mathbb{I} = \Theta$

$\Rightarrow$ $\mathbb{D}^3$ can be expressed in terms of $\mathbb{I}, \mathbb{D}, \mathbb{D}^2$ and so can $\mathbb{D}^n$ for any $n \in \mathbb{N}$, $n > 2$

$\Rightarrow$ the relation # can be simplified to

$$\mathbb{T}_D(\mathbb{D}) = \alpha_0 \mathbb{I} + \alpha_1 \mathbb{D} + \alpha_2 \mathbb{D}^2 \qquad \text{here } \alpha_i = \alpha_i({}_D I_1, {}_D I_2, {}_D I_3)$$

general relation for so-called Reiner-Rivlin fluids

if, in addition, this dependence is <u>linear</u>, then

$$\alpha_2 \equiv 0$$
$$\alpha_1 = \text{konst} = 2\mu$$
$$\alpha_0 = \alpha_0(\mathrm{Tr}\,\mathbb{D})$$

$$\Leftrightarrow \alpha_0 = \mu' \,\mathrm{Tr}\,\mathbb{D} = \mu' D_{ii} = \mu' \partial_i v_i = \mu' \nabla \cdot \vec{v}$$

$$\boxed{\begin{aligned} &\Rightarrow \mathbb{T}_D(\mathbb{D}) = \mu'(\nabla\cdot\vec{V})\,\mathbb{I} + 2\mu\,\mathbb{D} \\ &\Rightarrow \mathbb{T}(\mathbb{D}) = \left(-P + \mu'(\nabla\cdot\vec{V})\right)\mathbb{I} + 2\mu\,\mathbb{D} \end{aligned}}$$

Homework: write down $\mathbb{T}_D$ element-wise!

- Balance of linear momentum for Newtonian fluids = Navier-Stokes equations

- μ is the (coefficient of) dynamic viscosity
- μ' is the 2nd viscous coefficient

STOKES HYPOTHESIS

$$\mathbb{D} = \mathbb{D}_p + \mathbb{D}_{dev} \quad \text{where} \quad \mathbb{D}_p = \frac{1}{3}(\mathrm{Tr}\,\mathbb{D})\,\mathbb{I} = \frac{1}{3}(\nabla\cdot\vec{V})\,\mathbb{I}$$

"contribution to P"

deviatoric strain rate tensor

$$\mathbb{D}_{dev} = \mathbb{D} - \mathbb{D}_p = \frac{1}{2}\left[\left(\partial_j V_i - \partial_i V_j\right) - \frac{2}{3}\delta_{ij}\,\partial_k V_k\right]$$

$$\Rightarrow \quad \mathbb{T} = \left(-P + \mu'\nabla\cdot\vec{V}\right)\mathbb{I} + 2\mu\left(\mathbb{D}_p + \mathbb{D}_{dev}\right) = \qquad \mathrm{Tr}\,\mathbb{D}_{dev} = 0$$

$$= \underbrace{\left(-P + \left(\mu' + \frac{2}{3}\mu\right)\nabla\cdot\vec{V}\right)}_{\text{mechanical pressure}}\mathbb{I} + 2\mu\,\mathbb{D}_{dev}$$

mechanical pressure (generally different from the thermodynamic / hydrostatic pressure at rest denoted by P)

Stokes: $\underbrace{\mu' + \frac{2}{3}\mu}_{\varkappa} = 0 \Rightarrow \mu' = -\frac{2}{3}\mu$ but that's not true at all

$\varkappa$ ... (volumetric viscosity, bulk viscosity coefficient)

Buresti et al. 2015 $\quad \varkappa$ can be measured. For example, for CO_2 we have $\varkappa \gg 100\mu$

recall: $\frac{d}{dt}\int_{V(t)} \rho\phi\, d\vec{x} = \int_{V(t)} \rho \frac{D\phi}{Dt} d\vec{x}$

$$\boxed{\frac{D(\rho\vec{V})}{Dt} + \rho\vec{V}\nabla\cdot\vec{V} = -\nabla P + \nabla\cdot\Pi_D + \rho\vec{F}}$$

$\rho \frac{D\vec{V}}{Dt}$

linear momentum balance (= NS equations)

- INVISCID FLOW $\quad \mu\,(=\mu') = 0 \iff \Pi_D = 0$

$\Rightarrow$ Euler equations $\quad \boxed{\rho\frac{D\vec{V}}{Dt} = -\nabla P + \rho\vec{F}}$

- INCOMPRESSIBLE FLOW

– there are no (substantial) changes of volume during the flow

$$|V_0| = \underbrace{\int_{V_0} d\vec{X}}_{} \overset{!}{=} \int_{V(t)} d\vec{x} = \underbrace{\int_{V_0} |\det F|\, d\vec{X}}_{} = \text{const.}$$

$$\int_{V_0} (1 - |\det F|)\, d\vec{X} = 0 \quad \text{for any choice of } V$$

$$\Rightarrow |\det F| = 1 \quad \Rightarrow \quad \frac{\partial |\det F|}{\partial t} = |\det F|\, \nabla\cdot\vec{V} = 0$$

$$\Rightarrow \quad \nabla\cdot\vec{V} = 0 \quad \text{incompressibility condition}$$

What if $\rho = \text{konst}$?

continuity equation $\Big|\quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0 \quad \Rightarrow \quad \nabla\cdot\vec{V} = 0$

The other way round: $\quad \frac{\partial\rho}{\partial t} + \underbrace{\rho\nabla\cdot\vec{V}}_{=0} + \vec{V}\cdot\nabla\rho = 0$

$$\Rightarrow \quad \frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t}\Big|_{(t,\vec{X})} = 0$$

the initial density distribution is transported with the flow

If the material is <u>homogeneous</u> $\rho(0,\vec{X}) = \text{const} \Rightarrow \rho(t,\vec{x}) = \text{const} \quad \forall t, \forall \vec{x}$

In addition $\mu = \underbrace{\mu(\rho, T)}_{\text{generally}} = \text{const}$

Plug into NS equations

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\nabla P + \nabla\cdot\mathbb{T}_D + \rho\vec{F}}$$

and use $\boxed{\nabla\cdot\vec{V} = \partial_j V_j = 0}$

component-wise: $$\rho \frac{DV_i}{Dt} = -\partial_i P + \mu\,\partial_j\left(\partial_j V_i + \partial_i V_j\right) + \rho F_i$$

$$= -\partial_i P + \mu\,\partial_{jj} V_i + \mu\,\underbrace{\partial_{ij} V_j} + \rho F_i$$

$$= -\partial_i P + \mu\,\partial_{jj} V_i + \rho F_i \qquad \frac{\partial}{\partial_i}\underbrace{(\partial_j V_j)}_{=0}$$

$$\Downarrow$$

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \mu\Delta\vec{V} + \rho\vec{F}$$

$$\frac{D\vec{V}}{Dt} = -\nabla\tilde{P} + \nu\Delta\vec{V} + \vec{F}$$

$\nu = \frac{\mu}{\rho}$ .. kinematic viscosity

$\tilde{P} = \frac{P}{\rho}$ ... kinematic pressure

CONSERVATION OF ENERGY

Total energy contained in a control volume $\mathcal{V}(t) = \vec{x}(t, \mathcal{V}_0)$

$$\mathcal{E}(t) = \int_{\mathcal{V}(t)} \rho \left(E + \frac{1}{2} \vec{V}^2 \right) d\vec{x} \qquad \vec{V}^2 := \|\vec{V}\|^2 = \vec{V} \cdot \vec{V}$$

E ... specific internal energy (per unit mass)

Change of $\mathcal{E}$ per unit time is a sum of:

- power of surface forces

$$\int_{\partial \mathcal{V}(t)} \vec{V} \cdot (\mathbb{T} \cdot \vec{n}) \, dS = \int_{\partial \mathcal{V}(t)} V_i \tau_{ij} n_j \, dS$$

$$= \int_{\mathcal{V}(t)} \partial_j (V_i \tau_{ij}) \, d\vec{x}$$

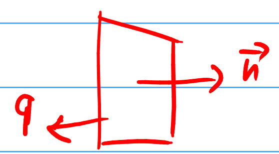
- power of volumetric forces

$$\int_{\mathcal{V}(t)} \vec{V} \cdot (\rho \vec{F}) \, d\vec{x} = \int_{\mathcal{V}(t)} \rho F_i V_i \, d\vec{x}$$

- flux of internal (heat) energy across $\partial \mathcal{V}(t)$ by heat conduction

Fourier's law : heat flux through unit surface with normal $\vec{n}$

is $\quad -\lambda \dfrac{\partial T}{\partial \vec{n}}$



heat conductivity $\quad \lambda = \lambda(T) \quad \left[\frac{W}{m^2} \right]$

increase of heat inside $\mathcal{V}(t)$

$$\int_{\partial \mathcal{V}(t)} \lambda \nabla T \cdot \vec{n} \, dS = \int_{\partial \mathcal{V}(t)} \lambda \partial_i T n_i \, dS = \int_{\mathcal{V}(t)} \partial_i (\lambda \partial_i T) \, d\vec{x}$$

$$= \int_{\mathcal{V}(t)} \nabla \cdot (\lambda \nabla T) \, d\vec{x}$$

- power of the volumetric heat socres (radiation)

$$\int_{V(t)} \varrho \dot{Q} \, d\vec{x} \qquad \dot{Q} \dots \text{ specific power of volum. heat sources}$$

Put together:

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt} \int_{V(t)} \varrho \overbrace{\left(E + \tfrac{1}{2}\vec{V}^2\right)}^{\phi} d\vec{x} = \int_{V(t)} \partial_j (V_i \tau_{ij}) + \varrho F_i V_i + \partial_i (\lambda \partial_i T) + \varrho \dot{Q} \, d\vec{x}$$

RTT: $\frac{d}{dt} \int_{V(t)} \varrho \phi \, d\vec{x} = \int_{V(t)} \varrho \frac{D\phi}{Dt} d\vec{x}$

$$\boxed{\varrho \frac{D(E + \frac{1}{2}\vec{V}^2)}{Dt} = \partial_j (V_i \tau_{ij}) + \varrho F_i V_i + \partial_i (\lambda \partial_i T) + \varrho \dot{Q}}$$

Conservation (balance) of total energy (COTE)

$\downarrow$

we express the balance of internal energy:

start from the linear momentum balance : $\varrho \frac{DV_i}{Dt} = \partial_j \tau_{ij} + \varrho F_i \qquad i = 1,2,3$

multiply the i-th component by V_i and (implicitly) sum over i

$$\varrho \underbrace{\frac{DV_i}{Dt} V_i}_{} = \partial_j \tau_{ij} V_i + \varrho F_i V_i$$

$$\downarrow$$

$$\varrho \frac{1}{2} \frac{D}{Dt} (\vec{V}^2)$$

this is subtracted from COTE

$$\Downarrow$$

$$\varrho \frac{DE}{Dt} = \tau_{ij} \partial_j V_i + \partial_i (\lambda \partial_i T) + \varrho \dot{Q}$$

$$\nabla \vec{V} = (\partial_j V_i)$$

(*) Note: (details later) $\frac{D}{Dt}\int_{T_{ref}}^{T} c_p(\theta)\,d\theta = c_p(T)\frac{DT}{Dt}$

in vector form: $$\boxed{\varrho\frac{DE}{Dt} = \mathbb{T}\cdot\nabla\vec{V} + \nabla\cdot(\lambda\nabla T) + \varrho\dot{Q}}$$

conservation of internal energy (COIE)

recap: 5 equations
7 unknowns $\varrho, V_1, V_2, V_3, P, T, E$

Remaining relations (for N-S eqs.) – 2 scalar (algebraic) equations – constitutive relations

1) equation of state
$$f(P,V,T) = f(\varrho,P,T) = 0$$

no universal laws of physics, but properties of the material

2) relation between E and T (*)

MATHEMATICAL ANALYSIS OF THE INCOMPRESSIBLE FLOW PROBLEM

Let $\Omega\subset\mathbb{R}^3$ (or $\Omega\subset\mathbb{R}^2$) be a bounded domain with Lipschitz boundary and $\mathcal{J}=(0,T_{max})$. The problem of incompressible flow inside Ω reads

$$(*)\qquad \nabla\cdot\vec{V} = 0$$ incompressibility (divergence-free) condition

$$(**)\qquad \frac{D\vec{V}}{Dt} = -\nabla\tilde{P} + \nu\Delta\vec{V} + \vec{F}$$

with Dirichlet b.c.
$$\vec{V}(t,\vec{x})\Big|_{\partial\Omega} = \vec{W}(t,\vec{x}) \qquad \forall\vec{x}\in\partial\Omega$$

where $\vec{W}$ satisfies
$$\int_{\partial\Omega}\vec{W}\cdot\vec{n}\,ds = 0$$

compatibility with divergence-free $\vec{V}$
$$\left\{ 0 \overset{!}{=} \int_{\Omega}\nabla\cdot\vec{V}\,d\vec{x} = \int_{\Omega}\partial_i V_i\,d\vec{x} = \int_{\partial\Omega}V_i n_i\,ds = \int_{\partial\Omega}\vec{V}\cdot\vec{n}\,ds \right.$$

and the initial condition
$$V(0,\vec{x}) = \vec{V}_0(\vec{x}) \quad \forall\vec{x}\in\Omega$$

hde $\quad \nabla\cdot\vec{V}_0(\vec{x}) = 0$

nontrivial to fulfill

What about the pressure field P ? (assuming sufficient regularity of $\vec{V}$)

take (**) and apply divergence on both sides of the equation:

$$\nabla \cdot \frac{D\vec{V}}{Dt} = -\Delta \tilde{P} + \nu \nabla \cdot \Delta \vec{V} + \nabla \cdot \vec{F}$$

denote $\quad \phi := \nabla \cdot \vec{V}$

LHS : $\nabla \cdot \frac{D\vec{V}}{Dt} = \nabla \cdot \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = \underbrace{\frac{\partial \phi}{\partial t}}_{=0 \text{ d/c } (*)} + \nabla \cdot \left(\vec{V} \cdot \nabla \vec{V} \right)$

RHS: $\nu \nabla \cdot \Delta \vec{V} = \nu \partial_i \partial_{kk} V_i = \nu \partial_{kki} V_i = \nu \partial_{ikk} V_i = \nu \partial_{kk} \partial_i V_i =$

$= \nu \Delta \phi$

plug back into (**) :

$$\Delta \tilde{P} + \nabla \cdot \left(\vec{V} \cdot \nabla \vec{V} \right) - \nabla \cdot \vec{F} = -\left(\frac{\partial \phi}{\partial t} - \nu \Delta \phi \right)$$

if (*) holds, i.e. $\phi = 0$, then

$$\boxed{\Delta \tilde{P} = -\nabla \cdot \left(\vec{V} \cdot \nabla \vec{V} \right) + \nabla \cdot \vec{F}}$$

(simplified) Pressure Poisson Equation (PPE, SPPE)

at each time instant, $\vec{P}$ depends (up to a constant) on $\vec{V}$ and $\vec{F}$
(a local change of $\vec{V}$ implies a global change of P)

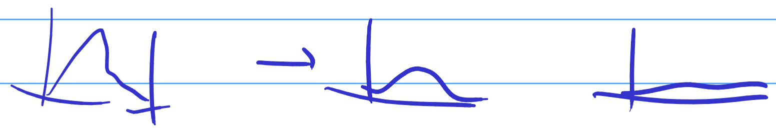
The other way round, if PPE holds, then $\underline{\frac{\partial \phi}{\partial t} = \nu \Delta \phi}$ "heat" eq. for ϕ

If $\nabla\cdot\vec{V}_0 = 0$, then $\phi(0,\vec{x}) = 0$. If we use such boundary conditions that $\phi\big|_{\partial\Omega} = 0$ or $\left.\dfrac{\partial\phi}{\partial\vec{n}}\right|_{\partial\Omega} = 0$,

then $\phi \equiv 0 \ \forall\vec{x}, \forall t$.

NOTE: In a numerical solver implementing the condition $\phi\big|_{\partial\Omega} = 0$, divergence of $\vec{V}$ will vanish over time even if $\nabla\cdot\vec{V}_0 \neq 0$

$\nabla\cdot\vec{V}(t)$



Alternative approach:

we leave this term here

$$\Delta\tilde{P} + \nabla\cdot\left(\vec{V}\cdot\nabla\vec{V}\right) - \nabla\cdot\vec{F} = -\left(\underbrace{\frac{\partial\phi}{\partial t}}_{=0} - \nu\Delta\phi\right)$$

$=0$ if (*) holds

$$\Rightarrow \quad \boxed{\Delta\tilde{P} = -\nabla\cdot\left(\vec{V}\cdot\nabla\vec{V}\right) + \nabla\cdot\vec{F} + \nu\nabla\cdot\left(\Delta\vec{V}\right)}$$

Consistent PPE, CPPE

Again, $\tilde{P}$ satisfying the consistent PPE implies $\dfrac{\partial\phi}{\partial t} = 0$

i.e. if $\phi = 0$ at $t = 0$, then $\phi \equiv 0 \ \forall t, \forall\vec{x}$ without any additional assumptions

Boundary conditions for $\tilde{P}$

Take (**) and multiply by $\vec{n}$ on $\partial\Omega$:

normal components of $\vec{V}, \vec{F}$

$$\left.\frac{D\vec{V}}{Dt} = -\nabla\tilde{P} + \nu\Delta\vec{V} + \vec{F}\ \right|\cdot\vec{n} \quad \Rightarrow \quad \frac{\partial\tilde{P}}{\partial\vec{n}} = -\frac{DV_n}{Dt} + \nu\Delta V_n + F_n$$

In the paper by Gresho & Sani (1987), it is shown that:

1) When this b.c. is used, both the simplified PPE and the consistent PPE are equivalent i.e. they both imply $\nabla\cdot\vec{V}=0$
2) for the CPPE, and for $t>0$, it is possible to apply a projection of (**) into directions other than $\vec{n}$ (i.e. tangential directions)

$V_i : \mathcal{T}\times\Omega \to \mathbb{R}$ $\quad V_i(t,\cdot):\Omega\to\mathbb{R}$

THE PATH TO THE WEAK FORMULATION

$V_i(t,\cdot)\in C^{(2)}(\Omega)$

$$\nabla\cdot\vec{V}=0 \;\wedge\; \underbrace{\frac{D\vec{V}}{Dt}}_{\left(\frac{\partial V_i}{\partial t}+V_j\partial_j V_i\right)\ \text{for } i=1,2,3} = -\nabla\tilde{P}+\nu\Delta\vec{V}+\vec{F}\ \text{\#)}$$

def (*)

$$\frac{\partial\vec{V}}{\partial t}+\underbrace{\vec{V}\cdot\nabla\vec{V}}_{①} = \underbrace{-\nabla\tilde{P}}_{②}+\underbrace{\nu\Delta\vec{V}}_{③}+\vec{F}_6$$

$\boxed{-\nabla(\tilde{P}-\phi)}$

multiply by a function $\vec{\varphi}:\Omega\to\mathbb{R}^3$ sufficiently smooth, equal to zero on $\partial\Omega$ and satisfying $\nabla\cdot\vec{\varphi}=0$. Then integrate over Ω

$$① = \int_\Omega \vec{V}\cdot\nabla\vec{V}\cdot\vec{\varphi}\,d\vec{x} = \int_\Omega V_j(\partial_j V_i)\varphi_i\,d\vec{x} = \text{(Green's formula)}$$

$$= \underbrace{\int_{\partial\Omega} V_jV_i\underbrace{\varphi_i}_{=0}n_j\,dS}_{=0} - \int_\Omega V_i\partial_j(V_j\varphi_i)\,d\vec{x} =$$

$$= -\int_\Omega V_i\underbrace{(\partial_j V_j)}_{=0}\varphi_i\,d\vec{x} - \int_\Omega V_i(\partial_j\varphi_i)V_j\,d\vec{x} = -\int_\Omega \vec{V}\cdot\nabla\vec{\varphi}\cdot\vec{V}\,d\vec{x}$$

#) $\vec{F}=\nabla\phi+\vec{F}_6$ where $\nabla\cdot\vec{F}_6=0$ (By Helmholtz theorem: $\vec{F}=\nabla\phi+\nabla\times\vec{A}$ if $\vec{F}\in C^2(\Omega)$)

$$② = \int_\Omega -\nabla\tilde{P}\cdot\vec{\varphi}\,d\vec{x} = -\int_\Omega(\partial_i\tilde{P})\varphi_i\,d\vec{x} \overset{\text{(Green)}}{=} -\int_{\partial\Omega}\tilde{P}\underbrace{\varphi_i}_{=0}n_i\,dS + \int_\Omega\tilde{P}\underbrace{\partial_i\varphi_i}_{=0}\,d\vec{x} = 0$$

$-\nabla(P-\phi)$

... ϕ doesn't play a role $\Rightarrow$ [WLOG $\phi=0 \Leftrightarrow \nabla\cdot\vec{F}=0$]

(Green)

$$③ = \nu \int_\Omega \Delta\vec{V}\cdot\vec{\varphi}\,d\vec{x} = \int_\Omega (\partial_{jj} V_i)\,\varphi_i\,d\vec{x} = \int_{\partial\Omega} \partial_j V_i\,\underbrace{\varphi_i}_{=0}\,n_j\,dS - \int_\Omega \partial_j V_i\,\partial_j \varphi_i\,d\vec{x}$$

$$= -\int_\Omega \nabla\vec{V} : \nabla\vec{\varphi}\,d\vec{x}$$

perform the integration of ⊛, using the above results:

(**)
$$\boxed{\underbrace{\int_\Omega \frac{\partial \vec{V}}{\partial t}\cdot\vec{\varphi}\,d\vec{x}}_{①} - \int_\Omega \vec{V}\cdot\nabla\vec{\varphi}\cdot\vec{V}\,d\vec{x} + \nu\int_\Omega \nabla\vec{V}\cdot\nabla\vec{\varphi}\,d\vec{x} = \int_\Omega \vec{F}\cdot\vec{\varphi}\,d\vec{x}}$$

this relation will come in handy later

\vartheta :-)

Now, take (**) and multiply by a smooth f. $\vartheta : \bar{\mathcal{J}} \to \mathbb{R}$ such that $\vartheta(T_{max}) = 0$ $\quad \mathcal{J} = (0, T_{max})$

and integrate over $\mathcal{J}$:

$$① = \int_{\mathcal{J}}\int_\Omega \frac{\partial \vec{V}}{\partial t}\cdot\vec{\varphi}\,d\vec{x}\,\vartheta\,dt = \int_{\mathcal{J}} \left(\frac{d}{dt}\underbrace{\int_\Omega \vec{V}\cdot\vec{\varphi}\,d\vec{x}}_{u}\right)\underbrace{\vartheta}_{v}\,dt = \left(\begin{matrix}\text{integration}\\ \text{by parts}\end{matrix}\right)$$

$$= \left[\int_\Omega \vec{V}\cdot\vec{\varphi}\,d\vec{x}\,\vartheta\right]_0^{T_{max}} - \int_{\mathcal{J}}\int_\Omega \vec{V}\cdot\vec{\varphi}\,d\vec{x}\,\dot{\vartheta}\,dt$$

$$= -\vartheta(0)\int_\Omega \vec{V}_0\cdot\vec{\varphi}\,d\vec{x} - \int_{\mathcal{J}}\int_\Omega \vec{V}\cdot\vec{\varphi}\,d\vec{x}\,\dot{\vartheta}\,dt$$

after integrating the whole (**):

$$\boxed{\int_{\mathcal{J}}\int_\Omega -\vec{V}\cdot\vec{\varphi}\,\dot{\vartheta} - \vec{V}\cdot\nabla\vec{\varphi}\cdot\vec{V}\vartheta + \nu\nabla\vec{V}\cdot\nabla\vec{\varphi}\,\vartheta - \vec{F}\cdot\vec{\varphi}\,\vartheta\,d\vec{x}\,dt = \vartheta(0)\int_\Omega \vec{V}_0\cdot\vec{\varphi}\,d\vec{x}}$$

WEAK EQUALITY (***)

OBSERVATIONS: → $\vec{V}$ and $\nabla\vec{V}$ must be defined ALMOST EVERYWHERE in Ω and A.E. in $\mathcal{Y}$

→ $\Delta\vec{V}$ need not exist

→ (***) does not contain the pressure $\tilde{P}$ at all

→ Any function $\vec{V}$ solving the original problem will also satisfy (***) for any $\vec{\varphi}$, $\mathcal{U}$

→ How to formulate a problem using (***) that would allow us to find $\vec{V}$?

- let us simplify the b.c. $\vec{V}|_{\partial\Omega} = \vec{W}$ hde $\int_{\partial\Omega} \vec{W}\cdot\vec{n}\,dS = 0$

to $\vec{V}|_{\partial\Omega} = \vec{0}$

SUMMARY OF KNOWN FUNCTION SPACES

- $C(\Omega)$ .. continuous functions $f : \Omega \to \mathbb{R}$
- $C^m(\Omega)$ ... space of functions with continuous m-th partial derivatives

def: $D^\alpha f := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} f$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$

multiindex

$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$

$$\Rightarrow C^m(\Omega) = \{ f : \Omega \to \mathbb{R} \mid D^\alpha f \in C(\Omega), |\alpha| \leq m \}$$

- $C_0^m(\Omega)$ .. Functions from $C^m(\Omega)$ with a compact support in Ω

$$\Rightarrow f \in C_0^m(\Omega) \Rightarrow f|_{\partial\Omega} = 0$$

closure is important! (e.g. continuous functions cannot be nonzero on a compact set)

$$\operatorname{supp}(f) = \overline{\{ \vec{x} \in \Omega \mid f(\vec{x}) \neq 0 \}}$$

$\partial\Omega \cap \Omega = \emptyset$

Ω is a domain $\Rightarrow$ $\Omega = \Omega^\circ$

$\Rightarrow \operatorname{supp}(f) \cap \partial\Omega = \emptyset$

- $L_m(\Omega)$ .. space of measurable functions such that $\int_\Omega |f(\vec{x})|^m d\vec{x} < +\infty$

Lebesgue integral

$m \in \langle 1, +\infty)$

$$\|f\|_{L_m(\Omega)} = \sqrt[m]{\int_\Omega |f(\vec{x})|^m d\vec{x}}$$

... $L_m(\Omega)$ is Banach (complete normed linear space)

→ $L_2(\Omega)$ is a Hilbert space with a scalar (inner) product

$$\langle f|g\rangle = \langle f, g\rangle = (f,g)_{L_2(\Omega)} = \int_\Omega f(\vec{x}) g(\vec{x}) d\vec{x}$$

$\|f\|_{L_2(\Omega)} \leq \|f\|_{H^m(\Omega)} \quad \forall f \in C^\infty(\bar{\Omega})$

$\Rightarrow (f_n)$ Cauchy in $\|\cdot\|_{H^m(\Omega)} \Rightarrow (f_n)$ Cauchy in $\|\cdot\|_{L_2(\Omega)}$

SOBOLEV SPACES

we add the limits of Cauchy seq., as they $\in L_2(\Omega)$

- $H^m(\Omega)$ is a closure (completion) of $C^\infty(\Omega)$ w.r.t the norm

$$\|f\|_{H^m(\Omega)} = \sqrt[2]{\int_\Omega \sum_{|\alpha| \leq m} (D^\alpha f)^2 d\vec{x}}$$

induced by the inner product

$$(f,g)_{H^m(\Omega)} = \sum_{\substack{\alpha \\ |\alpha| \leq m}} (D^\alpha f, D^\alpha g)_{L_2(\Omega)}$$

- $H^1_0(\Omega)$ ... functions from $H^1(\Omega)$ with compact support in Ω

NOTE : $H^m(\Omega) = \{ f : \Omega \to \mathbb{R} \mid D^\alpha f \in L_2(\Omega) \text{ pro } |\alpha| \leq m \}$

where $D^\alpha f$ denotes a weak derivative of f w.r.t. α, i.e., a regular distribution satisfying $\forall \varphi \in \mathcal{D}(\Omega)$

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi \, d\vec{x}$$

more general: Friedrichs ineq. for $f \in H^1(\Omega)$ for Ω with Lipschitz boundary $\quad \|f\|^2_{H^1(\Omega)} \leq k\left(\sum_{j=1}^{3}\int_\Omega |\partial_j f|^2 d\vec{x} + \int_{\partial\Omega} |f(\vec{x})|^2 dS\right)$ $\partial\Omega$ ← or $\Gamma \subset \partial\Omega$

$\exists k > 0$

- Poincare's inequality: $\quad \|f\|^2_{H^1_0(\Omega)} \leq k \underbrace{\sum_{j=1}^{3}\int_\Omega |\partial_j f|^2 d\vec{x}}_{\text{denote by } \|f\|'^2_{H^1_0(\Omega)}}$

"with derivatives of f bounded in Ω, the boundedness of f on $\partial\Omega$ implies its boundedness inside Ω"

- equivalence of $\|\cdot\|_{H^1_0(\Omega)}$ and $\|\cdot\|'_{H^1_0(\Omega)}$
- allows to estimate the L_2-norm of f by the L_2-norm of its derivatives

---

BOCHNER SPACES : Let $\mathcal{X}$ be a Banach (or Hilbert) space

Def: $$L_p(\mathcal{Y};\mathcal{X}) = \left\{ f : \mathcal{Y} \to \mathcal{X} \;\middle|\; \int_\mathcal{Y} \|f(t)\|^p_\mathcal{X}\, dt < +\infty \right\}$$

with the norm $$\|f\|_{L_p(\mathcal{Y};\mathcal{X})} = \sqrt[p]{\int_\mathcal{Y} \|f(t)\|^p_\mathcal{X}\, dt}$$

if $f : \mathcal{Y} \times \Omega \to \mathbb{R}$ then $f(t,\cdot) \in \{ w : \Omega \to \mathbb{R} \}$

FUNCTION SPACES SUITABLE FOR THE ANALYSIS OF INCOMP. FLOW

- $L_2(\Omega)^3$ Hilbert space of functions $\vec{u} : \Omega \to \mathbb{R}^3$ with components in $L_2(\Omega)$

$$\hookrightarrow (\vec{u},\vec{v})_{L_2(\Omega)^3} = \sum_{i=1}^{3} (u_i, v_i)_{L_2(\Omega)} = \sum_i \int_\Omega u_i v_i\, d\vec{x} = \int_\Omega \vec{u}\cdot\vec{v}\, d\vec{x}$$

$$\hookrightarrow \|\vec{u}\|_{L_2(\Omega)^3} = \Big\| \underbrace{\left(\|u_i\|_{L_2(\Omega)}\right)}_{\text{vector of norms}} \Big\| = \sqrt{\sum_i \|u_i\|^2_{L_2(\Omega)}} =$$

↑ Euclid. norm in $\mathbb{R}^3$

$$= \sqrt{\sum_i \int_\Omega u_i^2 d\vec{x}} = \sqrt{\int_\Omega \left(\sum_i u_i^2\right) d\vec{x}} = \sqrt{\int_\Omega \|\vec{u}\|^2 d\vec{x}}$$

- $H_0^1(\Omega)^3$ space of vector-valued func. $\vec{u}: \Omega \to \mathbb{R}^3$ with components in $H_0^1(\Omega)$

$$\|\vec{u}\|_{H_0^1(\Omega)^3} = \left\| \left(\|u_i\|_{H_0^1(\Omega)} \right) \right\|_{\mathbb{R}^3} = \sqrt{\sum_i \|u_i\|^2_{H_0^1(\Omega)}} =$$

$$= \sqrt{\sum_i \int_\Omega \sum_{j=1}^{3} (\partial_j u_i)^2 d\vec{x}} = \sqrt{\int_\Omega \underbrace{\nabla\vec{u} \cdot \nabla\vec{u}}_{\partial_j u_i \partial_j u_i} d\vec{x}}$$

NOTE: By Poincaré's inequality $\|\vec{u}\|_{L_2(\Omega)^3} \leq k \cdot \|\vec{u}\|_{H_0^1(\Omega)^3}$

- $C_{0,\sigma}^\infty(\Omega)^3$ ... vector-valued functions $\vec{u}: \Omega \to \mathbb{R}^3$ with components in $C_0^\infty(\Omega)$ satisfying in addition $\nabla \cdot \vec{u} = 0 \quad \forall \vec{x} \in \Omega$

- $H = L_{2,\sigma}(\Omega)^3$ closure (completion) of the space $C_{0,\sigma}^\infty(\Omega)^3$ in $L_2(\Omega)^3$

equipped with the norm of $L_2(\Omega)^3$

in this space, we have $\nabla \cdot \vec{u} = 0$ in the sense of distributions $\Rightarrow$ "weak" derivatives exist

- V ... closure of the space $C_{0,\sigma}^\infty(\Omega)^3$ in $H_0^1(\Omega)^3$

equipped with the norm of $H_0^1(\Omega)^3$

in this space, $\nabla \cdot \vec{u} = 0$ a.e.

ENERGY INEQUALITY

(a priori estimates of the (weak) solution of the problem)

Start from (**)

(**) :
$$\underbrace{\int_\Omega \frac{\partial \vec{v}}{\partial t} \cdot \vec{\varphi}\, d\vec{x}}_{①} - \underbrace{\int_\Omega \vec{v} \cdot \nabla\vec{\varphi} \cdot \vec{v}\, d\vec{x}}_{②} + \nu \int_\Omega \nabla\vec{v} \cdot \nabla\vec{\varphi}\, d\vec{x} = \int_\Omega \vec{F} \cdot \vec{\varphi}\, d\vec{x}$$

all terms in (**) allow to use $\vec{v}$ with $\nabla\cdot\vec{v}=0$ in place of $\vec{\varphi}$ (namely $\nabla\vec{\varphi}$ must exist a.e., but $\nabla\vec{v}$ appears in (**) anyway)

and put $\vec{\varphi}=\vec{v}$

$\vec{v}^2 = \vec{v}\cdot\vec{v} = \|\vec{v}\|^2_{\mathbb{R}^3}$

$$① = \int_\Omega \underbrace{\frac{\partial \vec{v}}{\partial t}\cdot\vec{v}}_{\sum_i(\partial_t v_i)v_i = \frac{1}{2}\sum\partial_t(v_i^2)}\,d\vec{x} = \int_\Omega \frac{1}{2}\frac{\partial \vec{v}^2}{\partial t}\,d\vec{x} = \frac{d}{dt}\underbrace{\int_\Omega \frac{1}{2}\vec{v}^2\,d\vec{x}}_{\text{specific kinetic energy in }\Omega}$$

②: some time ago, we derived that: $\int_\Omega \vec{v}\cdot\nabla\vec{v}\cdot\vec{\varphi}\,d\vec{x} = \cdots = -\int_\Omega \vec{v}\cdot\nabla\vec{\varphi}\cdot\vec{v}\,d\vec{x}$

we assumed that $\nabla\cdot\vec{\varphi}=0$ and also that $\vec{\varphi}|_{\partial\Omega}=0$

$$\Downarrow$$

$$② = \int_\Omega \vec{v}\cdot\nabla\vec{v}\cdot\vec{v}\,d\vec{x} = 0$$

$\Rightarrow$ Here the assumption $\underline{\vec{v}|_{\partial\Omega}=0}$ is important so that $\vec{v}$ can appear in place of $\vec{\varphi}$

plug ①, ② back to (**):

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\vec{v}^2\,d\vec{x} + \underbrace{\nu\int_\Omega \nabla\vec{v}\cdot\nabla\vec{v}\,d\vec{x}}_{\nu\|\vec{v}\|_V^2} = \underbrace{\int_\Omega \vec{F}\cdot\vec{v}\,d\vec{x}}_{(\vec{F},\vec{v})_{L_2(\Omega)^3}}$$

$$(\vec{F},\vec{v})_{L_2(\Omega)^3} \le \|\vec{F}\|_{L_2(\Omega)^3}\|\vec{v}\|_{L_2(\Omega)^3}$$

WLOG we assume $\nabla\cdot\vec{F}=0$ see above

$$\le k\underbrace{\|\vec{F}\|_H}_{b}\underbrace{\|\vec{v}\|_V}_{a}$$

Young's inequality

$$0 \le (\underbrace{a}_{\sqrt{\varepsilon}a} - \underbrace{b}_{\frac{1}{\sqrt{\varepsilon}}b})^2 = \varepsilon a^2 - 2ab + \frac{1}{\varepsilon}b^2$$

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$$

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

$\varepsilon = \frac{\nu}{k}$

$$\le \frac{\nu}{2}\|\vec{v}\|_V^2 + \frac{k^2}{2\nu}\|\vec{F}\|_H^2$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\vec{v}^2\,d\vec{x} + \frac{\nu}{2}\|\vec{v}\|_V^2 \le \frac{k^2}{2\nu}\|\vec{F}\|_H^2$$

$$\int \nabla V \cdot \nabla V \, d\vec{x} = -\int \underline{\Delta V}\, V \, d\vec{x}$$

integrate over $(0,t)$:

ENERGY INEQUALITY (usual in PDE analysis)

$$\underbrace{\int_\Omega \frac{1}{2}\vec{V}^2 dx \Big|_t}_{\text{kinetic energy of the fluid at time } t} + \underbrace{\int_0^t \frac{1}{2}\nu \|\vec{V}(\tau,\cdot)\|_V^2 d\tau}_{\text{dissipation of kinetic energy (diffusion of velocity)}} \le \underbrace{\int_\Omega \frac{1}{2}\vec{V}_0^2 d\vec{x}}_{\text{kinetic energy of the fluid at time } \underline{t=0}} + \frac{k^2}{2\nu}\underbrace{\int_0^t \|\vec{F}(\tau,\cdot)\|_H^2 d\tau}_{\text{work of volumetric forces from time } t=0 \text{ to } t}$$

1) if we neglect the second term on the LHS

$$\frac{1}{2}\|\vec{V}(t,\cdot)\|_H^2 \le \underbrace{\frac{1}{2}\|\vec{V}_0\|_H + \frac{1}{2}\frac{k^2}{\nu}\int_{\mathcal{Y}} \|\vec{F}\|_H^2 dt}_{\text{number independent of } t,\, \vec{x} \text{ and } \underline{\vec{V}}} \qquad \text{for almost all } \forall t \in \mathcal{Y}$$

$$\|\vec{V}\|^2_{L_\infty(\mathcal{Y},H)} \le \text{—"—}$$

$$\| \quad \operatorname*{ess\,sup}_{\mathcal{Y}} \|\vec{V}\|_H^2$$

2) if we neglect the first term on the LHS (and multiply by $\frac{2}{\nu}$)

... for $t = T_{max}$: $\|\vec{V}\|^2_{L_2(\mathcal{Y},V)} \le \frac{2}{\nu}\left(\text{—"—}\right)$

$\Rightarrow$ we get the boundedness of $\vec{V}$ in the spaces $\begin{cases} L_\infty(\mathcal{Y},H) \\ L_2(\mathcal{Y},V) \end{cases}$

norms of the

$$\Downarrow$$

see (*) above

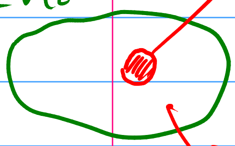
wouldn't $\vec{F} \in L_2(\mathcal{Y}, L_2(\Omega)^3)$ be enough?

DEF: **THE WEAK SOLUTION:** Let $\vec{V}_0 \in H$, $\vec{F} \in L_2(\mathcal{Y}, H)$

A function $\vec{V} \in L_\infty(\mathcal{Y}, H) \cap L_2(\mathcal{Y}, V)$ satisfying the weak equality

$$\boxed{\int_{\mathcal{Y}} \int_{\Omega} -\vec{V}\cdot\vec{\varphi}\,\dot{\nu} - \vec{V}\cdot\nabla\vec{\varphi}\cdot\vec{V}\nu + \nu\nabla\vec{V}\cdot\nabla\vec{\varphi}\,\nu - \vec{F}\cdot\vec{\varphi}\,\nu \, d\vec{x}\,dt = \nu(0)\int_{\Omega} \vec{V}_0\cdot\vec{\varphi}\,d\vec{x}}$$

$\forall \vec{\varphi} \in C^\infty_{0,\sigma}(\Omega)^3$ $\forall \nu \in C^\infty_0(\langle 0, T_{max}))$ is called the weak solution of the incompressible flow problem.



$\Omega = V(t')$

supp $\vec{\varphi}$ = $\mathcal{V}(t)$

$\vec{\varphi} = \vec{0}$

NOTE: The choice of φ, ν and performing $\int_\Omega$, resp. $\int_{\mathcal{Y}}$ corresponds to the choice of the control volume $\mathcal{V}_0$ and the time instant t in the integral form of the NSE

$\vec{\varphi} \simeq (1,1,1)^T$

"finite element method" vs. "finite volume method"

TOWARD THE PROOF OF EXISTENCE OF THE WEAK SOLUTION

NOTE: Galerkin's method (details later): $\vec{V} = \lim_{n\to+\infty} \vec{V}_n$ where $\vec{V}_n$ satisfies the weak equality only for $\vec{\varphi}$ from a finite-dimensional subspace of H $\Rightarrow$ We need to find a countable basis of H (not every infinite-dimensional space has one)

notion of "separable space"

STOKES OPERATOR

def $((\vec{u}, \vec{v})) = \int_\Omega \nabla\vec{u}\cdot\nabla\vec{v}\,d\vec{x}$ on V

bilinear form on V

The Stokes operator $A: V \to V'$ is defined by

$$\vec{u} \in V: \quad (A\vec{u})(\vec{v}) = ((\vec{u}, \vec{v})) = (\Delta\vec{u}, v)_H$$

in a sense, it's (more or less) the Laplacian

- Homework: prove that A is a linear operator

- A is bounded (= continuous)

$$\|A\vec{u}\|_{V'} = \sup_{\|\vec{v}\|_V=1} |A\vec{u}(\vec{v})| =$$

$$= \sup_{\|\vec{v}\|_V=1} |((\vec{u},\vec{v}))| =$$

$$= \sup_{\|\vec{v}\|_V=1} |(\vec{u},\vec{v})_V| \overset{\text{Schwarz}}{\leq}$$

$$\leq \sup_{\|\vec{v}\|_V=1} \|\vec{u}\|_V \underbrace{\|\vec{v}\|_V}_{=1} = \|\vec{u}\|_V$$

i.e. the constant $K=1$

- $A: \mathcal{B}_1 \to \mathcal{B}_1$ is continuous
$\Leftrightarrow$
$(\forall \varepsilon>0)(\exists\delta>0)(\|\vec{x}-\vec{y}\|<\delta)(\|A\vec{x}-A\vec{y}\|<\varepsilon)$
- A is bounded $\Leftrightarrow$
$(\exists K>0)(\|A\vec{x}\|_{\mathcal{B}_2} \leq K\|\vec{x}\|_{\mathcal{B}_1})$

... in Banach spaces, these two terms coincide

dual norm:

$$\|\underset{\leftarrow}{w}\|_{\mathcal{B}'} = \sup_{\|\vec{v}\|_{\mathcal{B}}=1} |\underset{\leftarrow}{w}(\vec{v})|$$

- $A: V \to V'$ is bijective and thus invertible

$((\cdot,\cdot))$ is coercive:

$$((\vec{u},\vec{u})) = \int_\Omega \nabla\vec{u}\cdot\nabla\vec{u}\,d\vec{x} = \|\vec{u}\|_V \geq k\,\|\vec{u}\|_{H_0^1(\Omega)^3}$$

but this is directly the inner product on V $\to$ $\mathcal{H}=V$ $\Rightarrow$ Lax-Milgram = Riesz

$\Rightarrow$ for each $\underset{\leftarrow}{w}\in V'$ $\exists_1 \vec{u}\in V$

so that $(\forall\vec{v}\in V$

$$\underset{\leftarrow}{w}(\vec{v}) = ((\vec{u},\vec{v})) = (A\vec{u})(\vec{v}))$$

$$\Leftrightarrow \boxed{\underset{\leftarrow}{w} = A\vec{u}}$$

<u>Note: Lax-Milgram lemma</u>

Let $\mathcal{H}$ be a Hilbert space with an inner product $(\cdot,\cdot)$.
$B: \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be a bilinear form satisfying

1) $B(\vec{u},\vec{v}) \leq K\|\vec{u}\|_{\mathcal{H}}\|\vec{v}\|_{\mathcal{H}}$ (boundedness)

2) $B(\vec{u},\vec{u}) \geq L\|\vec{u}\|^2_{\mathcal{H}}$ coercivity, $\mathcal{H}$-ellipticity

Then $\forall\underset{\leftarrow}{w}\in\mathcal{H}'$ $\exists_1\vec{u}\in\mathcal{H}$ such that
$\underset{\leftarrow}{w}(\vec{v}) = B(\vec{u},\vec{v})$.

In addition, $\|\vec{u}\|_{\mathcal{H}} \leq \frac{1}{L}\|\underset{\leftarrow}{w}\|_{\mathcal{H}'}$

- We know $V \subset H \Rightarrow H' \subset V'$ } in terms of boundedness w.r.t. the norm of H!

THEOREM: Let $w \in H'$. Then the unique solution of the equation
$$A\vec{u} = w$$
satisfies in addition

$H' \subset V'$.

$$\vec{u} \in H^2(\Omega)^3 \cap H_0^1(\Omega)^3 \cap L_{2,\sigma}(\Omega)^3 = H^2(\Omega)^3 \cap H_0^1(\Omega)^3 \cap V$$

i.e. the second derivatives exist and are L_2-bounded a.e. in Ω

We use the Riesz theorem (duality between H and H')

i.e. for each $w \in H'$ $\exists \vec{z} \in H$ such that $w(\vec{v}) = (\vec{z}, \vec{v})_H$

i.e. $\underbrace{(A\vec{u})}_{w}(\vec{v}) = (\vec{z}, \vec{v})_H = (-\Delta\vec{u}, \vec{v})_H$

$$\int_\Omega \nabla\vec{u} \cdot \nabla\vec{v}\, d\vec{x} = \int_\Omega \partial_j u_i \partial_j v_i\, dx \underset{\text{Green}}{=} -\int_\Omega \partial_{jj} u_i\, v_i\, dx = -\int_\Omega \Delta\vec{u} \cdot \vec{v}\, d\vec{x} = -(\Delta\vec{u}, \vec{v})_H$$

To sum up, we can understand A as an operator from D_A onto H

sometimes $A|_{D_A} \equiv \tilde{A}$ $\longrightarrow$ $\tilde{A}\vec{u} = -\Delta\vec{u}$

a restriction of the original operator A (this clarifies in what sense can A be considered a Laplacian)

i.e.

$H^2(\Omega)^3 \cap H_0^1(\Omega)^3 \cap V = D_A$ contains such elements that are mapped to $H' \equiv H$

- $\tilde{A}$ (in the sense $A: D_A \to H$) is bounded, invertible and also symmetric:

$$\boxed{(A\vec{u}, \vec{v})} = (-\Delta\vec{u}, \vec{v}) = ((\vec{u}, \vec{v})) = ((\vec{v}, \vec{u})) = (-\Delta\vec{v}, \vec{u}) = (A\vec{v}, \vec{u}) = \boxed{(\vec{u}, A\vec{v})}$$

$\Rightarrow$ symmetric + bounded = self-adjoint (over the field $\mathbb{R}$)

- $\forall A: \mathcal{H} \to \mathcal{H}$ $\exists A^*$ so that $\forall \vec{u}, \vec{v}$ we have $(A\vec{u}, \vec{v}) = (\vec{u}, A^*\vec{v})$. A^* is called the ADJOINT OP.
- $A = A^* \Rightarrow A$ is self-adjoint

$\Rightarrow \exists A^{-1}: H \to D_A$ which is also self-adjoint and COMPACT

Summary: $A: V \to V'$

$V \subset H \Rightarrow H \equiv H' \subset V'$

next, we need DEF: The operator $B: \mathcal{B}_1 \to \mathcal{B}_2$ is compact $\Leftrightarrow$ for each bounded set $M \subset \mathcal{B}_1$, the image $B(M)$ is a relatively compact set in $\mathcal{B}_2$.

i.e. $\overline{B(M)}$ is compact

DEF: Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces. We say that $\mathcal{B}_1$ is compactly embedded into $\mathcal{B}_2$ denoted by ($\mathcal{B}_1 \hookrightarrow\hookrightarrow \mathcal{B}_2$), iff $\mathcal{B}_1 \subset \mathcal{B}_2$ and the $\underbrace{\text{identity map}}_{\text{or "inclusion"}}$ $i: \mathcal{B}_1 \to \mathcal{B}_2$ is a compact operator.

so if $D_A \hookrightarrow\hookrightarrow H$, then A^{-1} is compact

THEOREM: Let $A: \mathcal{H}_1 \to \mathcal{H}_2$ be invertible, bounded and $\mathcal{H}_1 \hookrightarrow\hookrightarrow \mathcal{H}_2$. Then the inverse operator A^{-1} is compact.

THEOREM: A compact and self-adjoint operator B has a countable set of eigenvalues and the corresponding set of eigenvectors forms an orthogonal basis of $(\ker B)^\perp$.

then $B = A^{-1}: H \to D_A$ such that the eigenvectors of A^{-1} are an OG basis of $(\ker A^{-1})^\perp = (\{0\})^\perp = H$

THEOREM (Rellich-Kondrachov) Let $\Omega \subset \mathbb{R}^{3 (n)}$ be a bounded domain with a Lipschitz boundary. Then

a "Lipschitz domain"

$$W^{1,2}(\Omega) = \overset{V}{H^1(\Omega)} \hookrightarrow\hookrightarrow \overset{H \equiv H'}{L_2(\Omega)}$$

more generally: $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)$ for $1 \le q < \frac{np}{n-p} = \frac{3 \cdot 2}{3-2} = 6$

and

$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega)$ where $q = \frac{np}{n-p} (= 6)$

$W^{k,p}(\Omega)$ ... Sobolev space with $D^\alpha f \in L_p(\Omega)$ for $|\alpha| \le k$

"$\hookrightarrow$" ... continuous embedding, see below

$\Rightarrow$ by R-K. theorem, we indeed have $D_A \subset V \hookrightarrow\hookrightarrow H$ *) we prove this later

GALERKIN'S METHOD :

(we can always normalize an orthogonal basis if we like)

H has a countable basis $(\vec{W}_n)_{n=1}^{+\infty}$ which is orthonormal (ON)
WLOG

$$\Longleftrightarrow \quad (\vec{W}_k, \vec{W}_\ell)_H = \delta_{k\ell}$$

$\vec{W}_k$ is the eigenvector of $\tilde{A}^{-1}$

in addition, $((\vec{W}_k, \vec{W}_\ell)) = (A\vec{W}_k, \vec{W}_\ell)_H = (\mu_k \vec{W}_k, \vec{W}_\ell)_H = \mu_k \delta_{k\ell}$

$(\vec{W}_k, \vec{W}_\ell)_V$ $\qquad \int_\Omega \nabla \vec{W}_k \cdot \nabla \vec{W}_\ell \, d\vec{x}$

$\tilde{A}^{-1}\vec{W}_k = \lambda_k \vec{W}_k$

where $\lambda_k \neq 0$ (as $\tilde{A}^{-1}$ is bijective)

$\Rightarrow \tilde{A}\vec{W}_k = \frac{1}{\lambda_k}\tilde{A}\tilde{A}^{-1}\vec{W}_k = \frac{1}{\lambda_k}\vec{W}_k$

$\Rightarrow \vec{W}_k$ is also an eigenvector of $\tilde{A}$ with $\mu_k = \frac{1}{\lambda_k}$

Denote $\quad \underline{V_n} = \mathrm{span}(\vec{W}_1, \dots, \vec{W}_n)$

We seek the approximations of the weak solution $\vec{V}$ as

$$\vec{V}_n : \mathcal{Y} \to V_n$$

$$\vec{V}_n(t, \vec{x}) = \sum_{k=1}^{n} a_k(t) \vec{W}_k(\vec{x})$$

and that means $\vec{W}_k \in V$!

satisfying

this is again (**) with $\vec{V}_n$ instead of $\vec{V}$

$$\boxed{\int_\Omega \frac{\partial \vec{V}_n}{\partial t} \cdot \vec{\varphi}\, d\vec{x} - \int_\Omega \vec{V}_n \cdot \nabla \vec{\varphi} \cdot \vec{V}_n \, d\vec{x} + \nu \int_\Omega \nabla \vec{V}_n \cdot \nabla \vec{\varphi}\, d\vec{x} = \int_\Omega \vec{F} \cdot \vec{\varphi}\, d\vec{x}}$$

i.e. $$\sum_{k=1}^{n} \dot{a}_k(t) (\vec{W}_k, \vec{\varphi})_H - \sum_{k=1}^{n}\sum_{j=1}^{n} a_k(t) a_j(t) \int_\Omega \vec{W}_k \cdot \nabla \vec{\varphi} \cdot \vec{W}_j \, d\vec{x} + \nu \sum_{k=1}^{n} a_k(t) ((\vec{W}_k, \vec{\varphi})) = (\vec{F}, \vec{\varphi})_H$$

$\underline{\forall \vec{\varphi} \in V_n}$ $\qquad$ i.e. for $\underline{\vec{\varphi} = \vec{W}_\ell}$

$$(\vec{W}_k, \vec{\varphi})_H = (\vec{W}_k, \vec{W}_\ell)_H = \delta_{k\ell}$$

$$((\vec{W}_k, \vec{\varphi})) = \mu_\ell \delta_{k\ell}$$

$$\Rightarrow \boxed{\dot{a}_\ell(t) - \sum_{k=1}^{n}\sum_{j=1}^{n} a_k(t) a_j(t) \int_\Omega \vec{W}_k \cdot \nabla \vec{W}_\ell \cdot \vec{W}_j \, d\vec{x} + \nu \mu_\ell a_\ell(t) = (\vec{F}, \vec{W}_\ell)_H \quad \text{pro } \ell = 1, \dots, n}$$

a system of ODE's for $\vec{a}(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix}$

it is not like $\dot{\vec{a}}(t) = \vec{f}(t, \vec{a}(t))$, $\vec{f}$ does not explicitly depend on t

a system in the form $\dot{\vec{a}} = \vec{f}(\vec{a})$ is an autonomous system

$\Rightarrow$ from the theory of PDE's, such system has a solution either
- on the whole $\mathcal{Y} = (0, T_{max})$
- or on $(0, T_b)$ where $\lim_{t \to T_b-} \|\vec{a}\|_{\mathbb{R}^n} = +\infty$ blow-up

- we show the boundedness of $\|\vec{a}\|$ independently on time $\Rightarrow$ blow-up does not occur $\Rightarrow$ solution exists on $\mathcal{Y}$

NOTE: The initial conditions for the components a_ℓ ($\vec{a}(0) = \vec{a}_0$)

are given by $\vec{V}_0 = \sum_{n=1}^{+\infty} \beta_n \vec{W}_n$ (since $\vec{V}_0 \in H$)

as $a_\ell(0) = \beta_\ell \quad \forall \ell = 1, \dots, n$

which means that $\vec{V}_n(0) = \vec{V}_{0,n} = \sum_{k=1}^{n} \beta_k \vec{W}_k$

$$\|\vec{V}_n\|_H^2 = \int_\Omega \vec{V}_n \cdot \vec{V}_n \, d\vec{x} = (\vec{V}_n, \vec{V}_n)_H = \Big(\sum_{k=1}^n a_k \vec{W}_k, \sum_{j=1}^n a_j \vec{W}_j\Big)_H =$$

$$= \sum_{k=1}^n a_k \sum_{j=1}^n a_j \underbrace{(\vec{W}_k, \vec{W}_j)_H}_{\delta_{kj}} = \sum_{k=1}^n a_k^2 = \|\vec{a}\|^2_{\mathbb{R}^n}$$

$\Rightarrow$ we can estimate $\|\vec{V}_n\|_H^2$ instead of $\|\vec{a}\|_{\mathbb{R}^n}$

- we perform the same steps as in the derivation of the energy inequality. Starting from $(**)$ while having replaced $\vec{V}$ by $\vec{V}_n$, we plug in $\vec{\varphi} = \vec{V}_n$ and the rest is identical. (see above)

we obtain the a priori estimate

$$\Rightarrow \quad \|\vec{a}(t)\|_{R_n} = \|\vec{V}_n(t,\cdot)\|_H^2 \le \underbrace{\|\vec{V}_{0,n}\|_H + \frac{k^2}{\nu}\int_{\mathcal{T}} \|\vec{F}\|_H^2 \, dt}_{\text{a number independent of } t, \vec{x}} \quad \Big\} \Rightarrow \text{blow-up is not possible}$$

$\Rightarrow$ $\vec{V}_n$ exists on the whole $\mathcal{T}$

$$\Rightarrow \quad \|\vec{V}_n\|^2_{L_\infty(\mathcal{T},H)} \le -\!\!\shortparallel\!\!- \quad \Rightarrow \quad \vec{V}_n \in L_\infty(\mathcal{T},H)$$

... and also

$$\Rightarrow \quad \|\vec{V}_n\|^2_{L_2(\mathcal{T},V)} \le \frac{2}{\nu}\left(-\!\!\shortparallel\!\!-\right) \Rightarrow \vec{V}_n \in L_2(\mathcal{T},V)$$

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the story of embeddings & operators continues...

- $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ ($\mathcal{B}_1$ is continuously embedded into $\mathcal{B}_2$)

$\Leftrightarrow$ $i : \mathcal{B}_1 \to \mathcal{B}_2$ is continuous, i.e. bounded

$\Leftrightarrow$ $\|i\vec{v}\|_{\mathcal{B}_2} \le K \|\vec{v}\|_{\mathcal{B}_1}$

$\Leftrightarrow$ $\|\vec{v}\|_{\mathcal{B}_2} \le K \|\vec{v}\|_{\mathcal{B}_1}$

(simply said, the norm on $\mathcal{B}_2$ is dominated by the norm on $\mathcal{B}_1$

cf. "strong" convergence : $\vec{v}_n \to \vec{v}$ in $\mathcal{B} \iff \lim_{n\to+\infty} \|\vec{v}_n - v\|_{\mathcal{B}} = 0$

$\lim_{n\to+\infty} |\underset{\leftarrow}{w}(\vec{v}_n) - \underset{\leftarrow}{w}(\vec{v})| = 0$

D • $\vec{v}_n$ weakly converges to $\vec{v}$ in the Banach space $\mathcal{B}$

$\iff (\forall \underset{\leftarrow}{w} \in \mathcal{B}')(\underset{\leftarrow}{w}(\vec{v}_n) \to \underset{\leftarrow}{w}(\vec{v}))$ We denote $\underline{\vec{v}_n \rightharpoonup \vec{v}}$

D • The operator $T: \mathcal{B}_1 \to \mathcal{B}_2$ is COMPLETELY CONTINUOUS

$\iff \forall (\vec{v}_n)$ weakly convergent in $\mathcal{B}_1$, we have $T\vec{v}_n \to T\vec{v}$ in $\mathcal{B}_2$

i.e. $\|T\vec{v}_n - T\vec{v}\|_{\mathcal{B}_2} \to 0$

D • Banach $\mathcal{B}$ is REFLEXIVE $\iff$ $\mathcal{B}$ is isomorphic to its second dual $\mathcal{B}''$ "bidual"

$\Rightarrow$ by the Riesz representation theorem, every Hilbert space is reflexive

! Th. • In a reflexive Banach space, each bounded sequence has a weakly convergent subsequence. (this is an equivalent property)

Th. • Let $\mathcal{B}_1$ be reflexive and $T: \mathcal{B}_1 \to \mathcal{B}_2$ be completely continuous. Then T is also continuous (and bounded)

Th. • On a Banach space "T compact" $\Rightarrow$ "T completely continuous"

On a reflexive Banach space "T compact" $\iff$ "T completely continuous"

Th • $T: \mathcal{B}_1 \to \mathcal{B}_2$ continuous and invertible $\Rightarrow T^{-1}$ is continuous

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This is a good place to prove $V \subset\subset H$ by using the fact that both are Hilbert spaces, so the inclusion map $\imath: V \to H$ is compact $\iff$ it is completely continuous.

Lemma : $V \subset H$

Proof : We know $H_0^1(\Omega) \hookrightarrow\hookrightarrow L_2(\Omega)$

$\Rightarrow H_0^1(\Omega)^3 \subset L_2(\Omega)^3$ easy

$\left\{ \begin{array}{l} u_i \in H_0^1(\Omega) \Rightarrow u_i \in L_2(\Omega) \\ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in L_2(\Omega)^3 \end{array} \right.$

$V = \overline{C^1_{0,\sigma}(\Omega)}$ in $H_0^1(\Omega)^3$

$H = \overline{C^1_{0,\sigma}(\Omega)}$ in $L_2(\Omega)^3$

$\vec{u} \in V \Rightarrow \exists (\vec{u}_n) \subset C^1_{0,\sigma}(\Omega)$ such that $\vec{u}_n \to \vec{u}$ in $H_0^1(\Omega)^3$

i.e. $\|\vec{u}_n - \vec{u}\|_{H_0^1(\Omega)^3} \to 0$!

by definition of $\|\vec{u}\|^2_{H_0^1(\Omega)^3} = \sum_{i=1}^{3} \|u_i\|^2_{H_0^1(\Omega)^3}$

and by Poincare's inequality

$\|\vec{u}_n - \vec{u}\|_{L_2(\Omega)^3} \leq K \cdot \|\vec{u}\|^2_{H_0^1(\Omega)^3}$

$\Rightarrow \|\vec{u}_n - \vec{u}\|_{L_2(\Omega)^3} \to 0 \Longleftrightarrow \vec{u} \in H$

Theorem : $V \hookrightarrow\hookrightarrow H$

Proof : we know from Rellich-Kondrachov th. that

$H_0^1(\Omega) \hookrightarrow\hookrightarrow L_2(\Omega)$ } def #

that means by definition that $\imath : H_0^1(\Omega) \to L_2(\Omega)$ is compact, and hence completely continuous

1) We prove that $H_0^1(\Omega)^3 \hookrightarrow\hookrightarrow L_2(\Omega)^3$, i.e. by definition, that

$\imath : H_0^1(\Omega)^3 \to L_2(\Omega)^3$ is completely continuous, and hence compact

i.e. we want to prove that any weakly convergent sequence $(\vec{u}_n) \subset H_0^1(\Omega)^3$ converges strongly in $L_2(\Omega)^3$

Let $\vec{u}^n \rightharpoonup \vec{u}$ in $H_0^1(\Omega)^3$. So for any $\underleftarrow{w} \in H_0^1(\Omega)^{3\,\prime}$ we have

$\underleftarrow{w}(\vec{u}^n) \longrightarrow \underleftarrow{w}(\vec{u})$ $\quad \vec{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ where $u_i \in H_0^1(\Omega)$

$$\Rightarrow \vec{u}^n = \begin{pmatrix} u_1^n \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2^n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} \quad \text{and}$$

$$\underbrace{\underleftarrow{w}(\vec{u}^n) = \underleftarrow{w}\left(\begin{pmatrix} u_1^n \\ 0 \\ 0 \end{pmatrix}\right) + \underleftarrow{w}\left(\begin{pmatrix} 0 \\ u_2^n \\ 0 \end{pmatrix}\right) + \underleftarrow{w}\left(\begin{pmatrix} 0 \\ 0 \\ u_3^n \end{pmatrix}\right)}_{\text{linearity of } \underleftarrow{w}} = \sum_{k=1}^{3} \underbrace{\underleftarrow{w}_k}_{\underleftarrow{w}_k \in H_0^1(\Omega)'}(u_k^n)$$

and arbitrary choice of $\underleftarrow{w}$ implies as an arbitrary choice of $\underleftarrow{w}_k$

If we choose $\underleftarrow{w}_1$ ($k=1$) arbitrarily and $\underleftarrow{w}_2, \underleftarrow{w}_3 = \underleftarrow{0}$, then

$\underleftarrow{w}_1(u_1^n) \longrightarrow \underleftarrow{w}_1(u_1)$ and analogously for $k=2,3$

$\quad \shortparallel \qquad\qquad\quad \shortparallel$

$\underleftarrow{w}(\vec{u}^n) \longrightarrow \underleftarrow{w}(\vec{u})$

$\Rightarrow$ $u_k^n \rightharpoonup u_k$ for $k=1,2,3$. $\xrightarrow{(\#)}$ $u_k^n \to u_k$ in $L_2(\Omega)$
in $H_0^1(\Omega)$

$\Rightarrow$ $\underline{\|u_k^n - u_k\|_{L_2(\Omega)} \to 0 \quad \forall k=1,2,3}$

but $\underline{\|\vec{u}\|^2_{L_2(\Omega)^3} = \sum_k \|u_k\|^2_{L_2(\Omega)^3}}$

$\Downarrow$

$\Rightarrow$ $\|\vec{u}^n - \vec{u}\|_{L_2(\Omega)^3} \to 0$

$\Leftrightarrow$ $\vec{u}^n \to \vec{u}$ in $L_2(\Omega)^3$

$\Leftarrow$ $\quad \overset{\circ}{H}_1(\Omega)^3 \hookrightarrow\hookrightarrow L_2(\Omega)^3$

subset, but with the same norm

by definition of compact op.: if $V \subset \overset{\circ}{H}_1(\Omega)^3$

then $\mathcal{I}|_V : V \to L_2(\Omega)^3$ is also compact

and thus $V \hookrightarrow\hookrightarrow L_2(\Omega)^3$

but $V \subset H$

$\Rightarrow$ $V \hookrightarrow\hookrightarrow H$.

Bochner (but also Hilbert) space

We have $\vec{V}_n \in L_2(\mathcal{J}, V)$ $\Rightarrow$ reflexive $\Rightarrow$ $(\vec{V}_n)$ has a weakly convergent subsequence $(\vec{V}_{k_n})$

... from now on, denote simply $\vec{V}_{k_n} =: \vec{V}_n$

- We prove that the weak limit $\vec{V}$ $\left(\vec{V}_n \rightharpoonup \vec{V} \text{ in } L_2(\mathcal{J}, V)\right)$ is the weak solution of our problem

We multiply (**) with $\vec{V}_n$ in place of $\vec{V}$ by $\nu \in C_0^\infty(\langle 0, T_{max}))$, integrate over $\mathcal{J}$ and get an analog of the weak equality:

$$\boxed{\int_{\mathcal{J}}\int_{\Omega} \underbrace{-\vec{V}_n \cdot \vec{\varphi}\,\dot{\nu}}_{1} - \underbrace{\vec{V}_n \cdot \nabla \vec{\varphi} \cdot \vec{V}_n \nu}_{2} + \underbrace{\nu \nabla \vec{V}_n \cdot \nabla \vec{\varphi}\, \nu}_{3} - \vec{F} \cdot \vec{\varphi}\, \nu \, d\vec{x}\, dt = \nu(0) \int_{\Omega} \underbrace{\vec{V}_{0,n} \cdot \vec{\varphi}}_{4}\, d\vec{x}}$$

which is satisfied by "our" $\vec{V}_n$ $\quad \forall \vec{\varphi} \in V_n$, $\forall \nu \in C_0^\infty(\langle 0, T_{max}))$

let $\vec{\varphi} \in V_m$ where $m \in \mathbb{N}$ is fixed (!) and let $n \geq m$.

Obviously $V_m \subset V_n$

$$\iint 1 \, d\vec{x}dt = \int_{\mathcal{J}}\int_{\Omega} -\vec{V}_n \cdot \vec{\varphi}\,\dot{\nu}\, d\vec{x}\, dt \xrightarrow{n \to +\infty} \int_{\mathcal{J}}\int_{\Omega} -\vec{V} \cdot \vec{\varphi}\,\dot{\nu}\, d\vec{x}\, dt \quad \checkmark$$

as $\vec{V}_n \rightharpoonup \vec{V}$ in $L_2(\mathcal{J}, V)$

here: a linear functional $\in L_2(\mathcal{J}, V)'$ applied to $\vec{V}_n \in L_2(\mathcal{J}, V)$

and also here

$$\iint 3 \, d\vec{x}dt = \int_{\mathcal{J}}\int_{\Omega} \nu \nabla \vec{V}_n \cdot \nabla \vec{\varphi}\, \nu\, d\vec{x}\, dt \xrightarrow{n \to +\infty} \int_{\mathcal{J}}\int_{\Omega} \nu \nabla \vec{V} \cdot \nabla \vec{\varphi}\, \nu\, d\vec{x}\, dt \quad \checkmark$$

$$\iint 4 \, d\vec{x}dt = \nu(0) \int_{\Omega} \vec{V}_{0,n} \cdot \vec{\varphi}\, d\vec{x} \xrightarrow{n \to +\infty} \nu(0) \int_{\Omega} \vec{V}_0 \cdot \vec{\varphi}\, d\vec{x} \quad \checkmark$$

$$\vec{V}_{0,n} = \sum_{k=1}^{n} \beta_k \vec{W}_k \xrightarrow{n \to +\infty} \vec{V}_0$$

but what about the term 2 ? ... a nonlinear term w.r.t $\vec{V}_n$

$\Rightarrow$ weak convergence is not enough

V. (Lions-Aubin Lemma) Let $\mathcal{B}_0 \hookrightarrow\hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_1$.

$\mathcal{B}_0, \mathcal{B}_1$ be reflexive. Let $p_0, p_1 \in (1, +\infty)$ and define

$$Y = \left\{ \vec{u} \in L_{p_0}(\mathcal{J}, \mathcal{B}_0) \;\middle|\; \frac{\partial \vec{u}}{\partial t} \in L_{p_1}(\mathcal{J}, \mathcal{B}_1) \right\}$$

with the norm $\|\vec{u}\|_Y = \|\vec{u}\|_{L_{p_0}(\mathcal{J}, \mathcal{B}_0)} + \left\| \frac{\partial \vec{u}}{\partial t} \right\|_{L_{p_1}(\mathcal{J}, \mathcal{B}_1)}$

Then $Y \hookrightarrow\hookrightarrow L_{p_0}(\mathcal{J}, \mathcal{B})$.

The inner product $\left(\frac{\partial \vec{V}_n}{\partial t}(t,\cdot), \vec{\varphi} \right)_H \; \forall \vec{\varphi} \in V$ defines a map $\frac{\partial \vec{V}_n}{\partial t}(t,\cdot) : \vec{\varphi} \mapsto \left(\frac{\partial \vec{V}_n}{\partial t}(t,\cdot), \vec{\varphi} \right)_H$

$$\Rightarrow \frac{\partial \vec{V}_n}{\partial t}(t,\cdot) \in V' \quad \text{and thus} \quad \frac{\partial \vec{V}_n}{\partial t} : \mathcal{J} \to V'$$

$$\left\| \frac{\partial \vec{V}_n}{\partial t} \right\|_{L_{p_1}(\mathcal{J}, V')} = \left\| \left\| \frac{\partial \vec{V}_n}{\partial t}(t,\cdot) \right\|_{V'} \right\|_{L_{p_1}} = \left\| \sup_{\substack{\vec{\varphi} \in V \\ \|\vec{\varphi}\|_V = 1}} \left(\frac{\partial \vec{V}_n}{\partial t}(t), \vec{\varphi} \right)_H \right\|_{L_p}$$

$\leq \dots$ it is possible to estimate (give an upper bound)

$\left(p_1 = 2 \text{ for } \Omega \subset \mathbb{R}^2 \right.$

$\left. p_1 = \frac{4}{3} \text{ for } \Omega \in \mathbb{R}^3 \right)$

Lions-Aubin

$\Rightarrow p_0 = 2$ +

$\mathcal{B}_0 = V, \; \mathcal{B} = H, \; \mathcal{B}_1 = V'$

in addition $V \hookrightarrow\hookrightarrow H \hookrightarrow V'$ $\Leftarrow$ $\begin{cases} V \hookrightarrow\hookrightarrow H \Rightarrow V \hookrightarrow H \Rightarrow H' \hookrightarrow V' \\ H \equiv H' \Rightarrow H \hookrightarrow V' \end{cases}$

at the same time, we do NOT use $V \equiv V'$ { that's in a different norm

$$Y$$
$$\parallel$$

$$\Rightarrow L_2(\gamma; V) \hookrightarrow\hookrightarrow L_2(\gamma; H)$$

$\Rightarrow$ each weakly convergent sequence in $L_2(\gamma, V)$ converges strongly in $L_2(\gamma, H)$

i.e. $\vec{V}_n \rightharpoonup \vec{V}$ in $L_2(\gamma, V)$ $\Rightarrow$ $\vec{V}_n \to \vec{V}$ in $L_2(\gamma, H)$

$\Rightarrow$ back to the term 2

$$\left| \iint_{\gamma\Omega} \vec{V}_n \cdot \nabla\vec{\varphi} \cdot \vec{V}_n \, \mu \, d\vec{x}\, dt - \iint_{\gamma\Omega} \vec{V} \cdot \nabla\vec{\varphi} \cdot \vec{V} \, \mu \, d\vec{x}\, dt \right| =$$

$$\left| \iint_{\gamma\Omega} \Big(\underbrace{\vec{V}_n \cdot \nabla\vec{\varphi} \cdot \vec{V}_n - \vec{V} \cdot \nabla\vec{\varphi} \cdot \vec{V}_n}_{A} + \underbrace{\vec{V} \cdot \nabla\vec{\varphi} \cdot \vec{V}_n - \vec{V} \cdot \nabla\vec{\varphi} \cdot \vec{V}}_{B} \Big) \mu \, d\vec{x}\, dt \right|$$

$$\le \underbrace{\iint_{\gamma\Omega} |A| \, d\vec{x}\, dt}_{\text{will be treated below}} + \underbrace{\iint_{\gamma\Omega} |B| \, d\vec{x}\, dt}_{\iint \left| \vec{V} \cdot \nabla\vec{\varphi} \cdot (\vec{V}_n - \vec{V}) \mu \right| d\vec{x}\, dt}$$

again, just a linear functional applied to $\vec{V}_n - \vec{V}$

i.e. it is enough to have $\vec{V}_n \rightharpoonup \vec{V}$ in order to $(\vec{V}_n - \vec{V}) \rightharpoonup \vec{0}$

$$\Rightarrow \iint_{\gamma\Omega} |B| \, d\vec{x}\, dt \to 0$$

$$\iint_{\gamma\Omega} |A|\, d\vec{x}\, dt = \iint_{\gamma\Omega} \Big|\Big(\underbrace{\vec{V}_n \cdot \nabla\vec{\varphi} \cdot \vec{V}_n - \vec{V} \cdot \nabla\vec{\varphi} \cdot \vec{V}_n}_{A} \Big)\Big| \mu \; d\vec{x}\, dt =$$

Hölder's inequality

$$= \iint_{\gamma\Omega} \Big| (\vec{V}_n - \vec{V}) \cdot \nabla\vec{\varphi} \cdot \vec{V}_n \, \mu \Big| \, d\vec{x}\, dt \le$$

$$\sum_{n}^{N} a_n b_n \le \left(\sum |a_n|^p \right)^{1/p} \left(\sum |b_n|^q \right)^{1/q}$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\int_\Omega f g \, d\vec{x} \le \left(\int |f|^p dx \right)^{1/p} \left(\int |g|^q dx \right)^{1/q}$$

$$\int \vec{f} \cdot \vec{g} \, d\vec{x} = \int \sum_i f_i g_i \, d\vec{x} \le \quad p = q = 2$$

$$\le \int \underbrace{\sqrt{\sum f_i^2}}\, \cdot \underbrace{\sqrt{\sum g_i^2}} \, d\vec{x}$$

$$\le \sqrt{\int \|\vec{f}\|^2 dx} \cdot \sqrt{\int \|g\|^2 dx}$$

$$\le \underbrace{\sqrt{\iint_{\gamma\Omega} \|\vec{V}_n - \vec{V}\|^2 d\vec{x}\, dt}} \; \underbrace{\sqrt{\iint_{\gamma\Omega} \|\nabla\vec{\varphi} \cdot \vec{V}_n\|^2 \mu^2 \, d\vec{x}\, dt}}$$

$$(A1) = \|\vec{V}_n - \vec{V}\|_{L_2(\gamma; H)} \to 0$$

it is enough to prove this is bounded (A2)

$$(A2) = \int_\Omega \|\nabla\vec{\varphi} \cdot \vec{V}_n\| \, d\vec{x} = \int_\Omega \sum_{j=1}^{3} \Big(\underbrace{\sum_{i=1}^{3} \partial_j \varphi_i V_{n,i}}_{\text{Hölder } p=q=\frac{1}{2}} \Big)^2 d\vec{x} \le$$

$$\le \int_\Omega \sum_{j=1}^{3} \left[\sum_{i=1}^{3} (\partial_j \varphi_i)^2 \cdot \sum_{i=1}^{3} V_{n,i}^2 \right] d\vec{x} = \int_\Omega \underbrace{(\nabla\vec{\varphi} \cdot \nabla\vec{\varphi})}_{\text{scalar}} \underbrace{\|\vec{V}_n\|^2}_{\text{scalar}} d\vec{x} \le$$

$$\text{Hölder}: \quad \frac{1}{p} = \frac{1}{3}, \quad \frac{1}{q} = \frac{2}{3} \iff p = 3, \; q = \frac{3}{2}$$

For this to be bounded, $\vec{\varphi} \in V_m$ is not enough

$\Rightarrow$ we want

$\vec{\varphi} \in V_m \cap C_{0,\sigma}^{\infty}(\Omega)^3$!

$$\leq \underbrace{\left(\int_{\Omega} \|\vec{v}_n\|^6 \, dx \right)^{1/3}} \cdot \left(\int_{\Omega} (\nabla\vec{\varphi} \cdot \nabla\vec{\varphi})^{3/2} \, d\vec{x} \right)^{2/3}$$

$\vec{v}_n \in V = H_0^1(\Omega)^3 \cap L_{2\sigma}(\Omega)^3 \Rightarrow$ Rellich-Kondrachov

$\|\vec{v}_n\|_{L_6(\Omega)} \leq K \cdot \|\vec{v}_n\|_V$ ✓

$\Rightarrow \vec{v}$ satisfies the weak inequality $\forall \vec{\varphi} \in V_m \cap C_{0,\sigma}^{\infty}(\Omega)^3 \quad \forall m \in \mathbb{N}$

$\Rightarrow$ for $m \to +\infty$

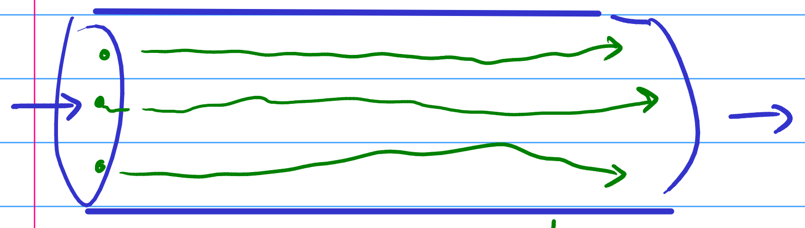
warning .. closure argument .. (limit elements missing)

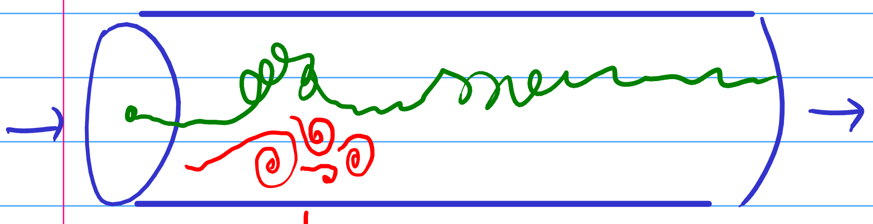
$$\forall \vec{\varphi} \in \bigcup_{m=1}^{+\infty} V_m \cap C_{0,\sigma}^{\infty}(\Omega)^3 = H \cap C_{0,\sigma}^{\infty}(\Omega)^3 = C_{0,\sigma}^{\infty}(\Omega)^3$$

contains all finite combinations of (w_n)

REMARKS:
- uniqueness of the solution (2D $\Rightarrow$ OK, 3D $\Rightarrow$?)
- satisfying the energy inequality ? ("our" $\vec{v}$ is OK)
- existence of pressure field P
  - (Pokorný - NS course - MFF)
- existence of smooth solutions
  - (Ladyženská, Kyselev 1957 - local in time solutions)
- analysis of compressible viscous flow ...

TURBULENT FLOW





- the onset of vortices $\Rightarrow$ chaotic flow

the { faster flow, lower viscosity, wider tube }, the more chaos

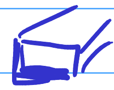
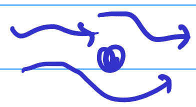
Reynolds number $$Re = \frac{\rho |\vec{V}| L}{\mu} = \frac{|\vec{V}| L}{\nu}$$

L ... charact. dimension

μ ... dyn. viscosity

ν ... kinematic viscosity

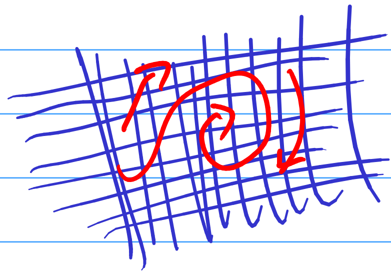
L ...

-  ... hydraulic radius  r $\frac{ab}{a+b}$ 
-  ... dimension of the obstacle in the flow

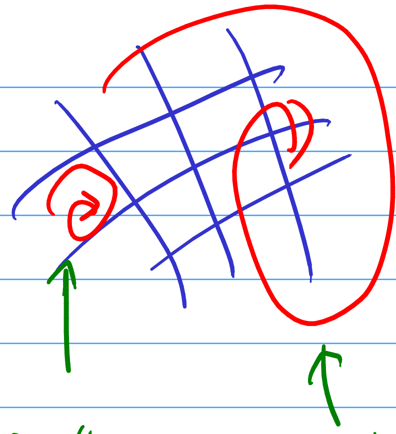
$Re < 1000$ ... laminar flow

$\gg 1000$ ... turbulent flow

NMDT :



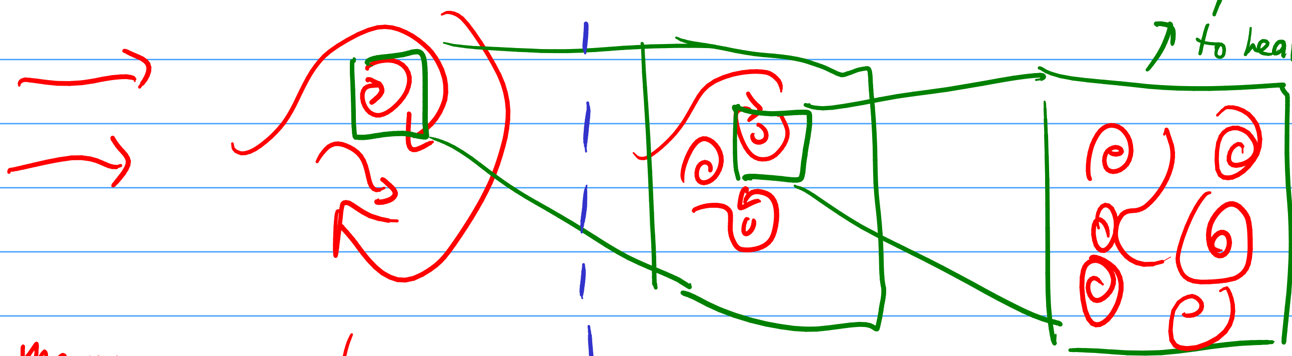
numerical mesh fine enough to "resolve" the vortices in the flow



small eddies cannot be resolved

the bigger ones can

dissipation ↗ to heat



macro scale

large eddies

smaller

even smaller

can be represented on a mesh

cannot be represented

Kolmogorov scale

⇒ a MODEL is needed

How to model turbulence

1) Reynolds averaging

- in time (T)
- in space (V)
- over statistical ensemble (N)

$$f = \bar{f} + f'$$

quantity : $\in \{\Theta, V_i, P\}$

average value

fluctuation

$$\bar{f}_T(t,\vec{x}) = \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} f(\tau,\vec{x})\,d\tau$$

$T \to +\infty$ for stationary flow

$$\bar{f}_{\mathcal{V}}(t,\vec{x}) = \lim_{|\mathcal{V}|\to+\infty} \frac{1}{|\mathcal{V}|}\int_{\mathcal{V}(\vec{x})} f(t,\vec{\xi})\,d\vec{\xi}$$

(homogeneous flow)

$$\bar{f}_N(t,\vec{x}) = \lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} f^{(n)}(t,\vec{x})$$

$\leftarrow$ number of experiments

"$\to +\infty$" = ..., the temporal/spatial scale of T, $|\mathcal{V}|$ is at least an order of magnitude larger than that of turbulent flow features

<u>The Reynolds averaging rules</u>

$$\overline{f+g} = \bar{f} + \bar{g}\,, \qquad \overline{\alpha f} = \alpha \bar{f}$$

$$\overline{fg} = \bar{f}\cdot\bar{g} + \overline{f'g'}$$

$$\overline{\partial_i f} = \partial_i \bar{f}$$

time averaging

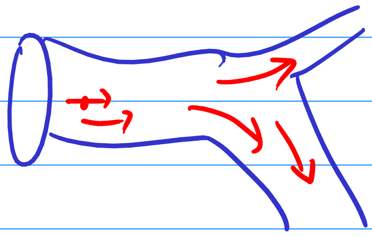
$$\bar{\bar{f}}(t,\vec{x}) = \lim_{T\to+\infty} \int_t^{t+T} \bar{f}(\tau,\vec{x})\,dt = \bar{f}(t,\vec{x})$$

$$\bar{\bar{f}} = \bar{f} \Rightarrow \bar{f'} = \overline{(f-\bar{f})} = \bar{f} - \bar{\bar{f}} = \bar{f} - \bar{f} = 0$$

For non-stationary flow, these relations are satisfied <u>asymptotically</u>.

for $T \not\to +\infty$ but $T = \delta t$, where $\frac{\delta t}{t_0} \to 0$

time scale of the macroscopic phenomena



Plug all quantities expressed as

$$\rho = \bar{\rho} + \rho'$$
$$V_i = \bar{V}_i + V_i'$$
$$p = \bar{p} + p'$$

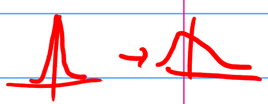
into the NS equations (for incompressible flow)

$$\Rightarrow \quad \frac{\partial \bar{\rho}}{\partial t} + \partial_i \left(\bar{\rho} \bar{V}_i \right) = 0$$

$$\frac{D\bar{V}_i}{Dt} = -\partial_i \bar{p} + \mu \Delta \bar{V}_i + \partial_j \left(\tau^R_{ij} \right) + F_i$$

the same form as the original NS eq.

T^R introduces TURBULENT VISCOSITY into the flow



$$\left(\tau^R_{ij} \right) = \mathbb{T}^R \;\ldots\; \text{Reynolds stress tensor}$$

$$t^R_{ij} = \overline{V_i' V_j'} = \bar{V}_i \bar{V}_j - \overline{V_i V_j} \quad \ldots \text{ is symmetric}$$

$$\frac{1}{2} \mathrm{Tr}\, \mathbb{T}^R = \frac{1}{2} \overline{V_i' V_i'} = \frac{1}{2} \overline{\sum V_i'^2} \quad \ldots \text{ average TURBULENT KINETIC ENERGY}$$

created from kin. energy $\sum_i \frac{1}{2} \left(\bar{V}_i \right)^2$

transported with the flow at velocity $\bar{\vec{V}}$

dissipates to heat

T^R has to be modeled ($\Rightarrow$ closure)

1st order models — algebraic models / one-equation models / two-equation models

linear dependence of T^R on the "mean stress tensor" S

... most common models are 1st order

Spalart-Allmaras one-equation model

$$
\begin{aligned}
\frac{\partial \tilde{\nu}}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\nu} v_j) = {} & C_{b1}(1 - f_{t2})\tilde{S}\tilde{\nu} \\
& + \frac{1}{\sigma}\left\{\frac{\partial}{\partial x_j}\left[(\nu_L + \tilde{\nu})\frac{\partial \tilde{\nu}}{\partial x_j}\right] + C_{b2}\frac{\partial \tilde{\nu}}{\partial x_j}\frac{\partial \tilde{\nu}}{\partial x_j}\right\} \\
& - \left[C_{w1} f_w - \frac{C_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + f_{t1}\|\Delta \vec{v}\|_2^2 .
\end{aligned}
$$

turbulent viscosity, "eddy viscosity"

$$
\begin{aligned}
\tilde{S} &= f_{v3}\, S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}, \\
f_{v1} &= \frac{\chi^3}{\chi^3 + C_{v1}^3}, \quad f_{v2} = \left(1 + \frac{\chi}{C_{v2}}\right)^{-3}, \\
f_{v3} &= \frac{(1 + \chi f_{v1})(1 - f_{v2})}{\max(\chi, 0.001)}, \quad \chi = \frac{\tilde{\nu}}{\nu_L}.
\end{aligned}
$$

DESTRUCTION OF $\tilde{\nu}$

$$
\begin{aligned}
f_w &= g\left(\frac{1 + C_{w3}^6}{g^6 + C_{w3}^6}\right)^{1/6}, \\
g &= r + C_{w2}(r^6 - r), \quad r = \frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}.
\end{aligned}
$$

PRODUCTION OF $\tilde{\nu}$

$$
S = \sqrt{2\,\Omega_{ij}\Omega_{ij}},
$$

$$
\Omega_{ij} = \frac{1}{2}\left(\partial v_i / \partial x_j - \frac{\partial v_j}{\partial x_i}\right).
$$

skew-symmetric part of the velocity gradient (rate of rotation)

- two-equation models (e.g., K - ε model)

turbulent kinetic energy

rate of dissipation of K

$$\frac{\partial \rho K}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j K) = \frac{\partial}{\partial x_j}\left[\left(\mu_L + \frac{\mu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_j}\right] + \tau_{ij}^F S_{ij} - \rho\varepsilon$$

$$\frac{\partial \rho \varepsilon^*}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j \varepsilon^*) = \frac{\partial}{\partial x_j}\left[\left(\mu_L + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon^*}{\partial x_j}\right] + C_{\varepsilon 1} f_{\varepsilon 1}\frac{\varepsilon^*}{K}\tau_{ij}^F S_{ij} - C_{\varepsilon 2} f_{\varepsilon 2}\,\rho\,\frac{(\varepsilon^*)^2}{K} + \phi_\varepsilon .$$

$$\varepsilon = \varepsilon_w + \varepsilon^* . \qquad \mu_T = C_\mu f_\mu\, \rho\, \frac{K^2}{\varepsilon^*}.$$

total dissipation of t.k.e.

dissipation at the wall (boundary condition for ε)

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92,$$
$$\sigma_K = 1.0, \quad \sigma_\varepsilon = 1.3, \quad Pr_T = 0.9.$$

Furthermore, the near-wall damping functions read

$$\begin{aligned} f_\mu &= \exp\left(\frac{-3.4}{(1 + 0.02\, Re_T)^2}\right) \\ f_{\varepsilon 1} &= 1 \\ f_{\varepsilon 2} &= 1 - 0.3 \exp\left(Re_T^2\right) \end{aligned} \tag{7.54}$$

with $Re_T = \rho K^2/(\varepsilon^* \mu_L)$ being the turbulent Reynolds number. Finally, the explicit wall term ϕ_ε and the value ε_w are defined as

$$\phi_\varepsilon = 2\mu_T \frac{\mu_L}{\rho}\left(\frac{\partial^2 v_s}{\partial y_n^2}\right)^2 \quad \text{and} \quad \varepsilon_w = \frac{2\mu_L}{\rho}\left(\frac{\partial \sqrt{K}}{\partial y_n}\right)^2, \tag{7.55}$$

SOURCE : COMPUTATIONAL FLUID DYNAMICS
Principles and Applications
Third Edition

JIRI BLAZEK, PhD
CFD Consulting & Analysis
Sankt Augustin, Germany

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ELEMENTS OF THERMODYNAMICS OF FLUIDS

- compressible flow : continuity eq + 3x NS eq + energy eq. = 5 equations

  $\underbrace{V_i}_{\times 3}, P, \varrho, T, E$ = 7 unknowns

RELATIONSHIP BETWEEN T and E

Specific heat capacities

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V \qquad \text{resp.} \quad c_p = \left(\frac{\partial H}{\partial T}\right)_P$$

where $H = E + \frac{P}{\varrho}$ ... specific enthalpy

REMARK : Denote $\mathcal{E}(T) = E(\phi(T))$ where $\phi(T) = \underline{(P(T), V, T)}$

$V = \text{const}$

$P = P(T)$

then $c_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{d\mathcal{E}}{dT} = \ldots$

NOTE : In liquids, usually $\varrho = \text{const}$ (close to incompressible)

$$\Rightarrow \quad c_p = \left(\frac{\partial H}{\partial T}\right)_P = \left(\frac{\partial E}{\partial T}\right)_P + \underbrace{\left(\frac{\partial \left(\frac{P}{\varrho}\right)}{\partial T}\right)_P}_{\approx 0} \approx \left(\frac{\partial E}{\partial T}\right)_P$$

$P = \text{const.}$

$\varrho \approx \text{const.}$

Generally, $c_v = c_v(T)$, $c_p = c_p(T)$ but if temperature range is narrow $\Rightarrow c_v = c_p \approx$ const. $\Rightarrow$ $E = \int_{T_0}^{T} \underbrace{c_{p,v}}_{\text{const.}}(\tau)\,d\tau = c_{p,v}\cdot(T-T_0)$

Energy eq. contains T only in derivatives.

$\Rightarrow$ the choice of T_0 is not relevant

$$\Rightarrow \frac{DE}{Dt} = c_{p,v} \cdot \frac{DT}{Dt}$$

$c_{p,v}(T)$

<u>EQUATION OF STATE</u> : relationship $f(P,T,V) = \tilde{f}(P,T,\varrho) = 0$

in terms of specific quantities

- EOS of ideal gas

$$PV = nRT$$

volume, molar amount

R .. universal gas constant $\approx 8{,}314..\ \mathrm{J\cdot mol^{-1}K}$

$$P = \frac{n}{V}RT = \frac{\overset{m}{nM}}{V}\frac{R}{M}T = \varrho R_{spec} T$$

ϱ

specific gas constant

$R_{spec} = \frac{R}{M}$

M... molar mass of the substance

ideal gas EOS $\Leftarrow$ single-atom molecules without mutual interactions

$\Rightarrow$ works well for dilute gases, but not for
- liquids
- gases close to boiling temperature
- saturated vapors

ALTERNATIVE (more accurate) EOS

• virial EOS

$$z = \frac{pV_\mathrm{m}}{\boldsymbol{R}T} = 1 + \frac{B}{V_\mathrm{m}} + \frac{C}{V_\mathrm{m}^2} + \ldots = \frac{p}{\boldsymbol{R}T\rho} = 1 + B\rho + C\rho^2 + \ldots$$

2. 3. virial coefficient

(they are functions of temperature only)

• virial EOS for mixtures

$$B = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j B_{ij}$$

mutual interactions of molecules of the i-th and j-th mixture components

$$C = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N} \underline{x_i x_j x_\ell} C_{ij\ell}$$

mutual interaction of 3 different molecules ... etc.

concentration of the (i,j,ℓ)-th component

- Van der Waals EOS

$$p = \frac{n\boldsymbol{R}T}{V - nb} - \frac{an^2}{V^2} = \frac{\boldsymbol{R}T}{V_\mathrm{m} - b} - \frac{a}{V_\mathrm{m}^2}$$

$$p = \frac{\boldsymbol{R}T\rho}{1 - b\rho} - a\rho^2, \qquad z = \frac{p}{\boldsymbol{R}T\rho} = \frac{1}{1 - b\rho} - \frac{a\rho}{\boldsymbol{R}T}$$

$$a = \frac{27}{64}\frac{\boldsymbol{R}^2T_\mathrm{c}^2}{p_\mathrm{c}}, \qquad b = \frac{1}{8}\frac{\boldsymbol{R}T_\mathrm{c}}{p_\mathrm{c}}$$

← p_c, T_c ... conditions at the triple point

- Redlich – Kwong EOS

$$p = \frac{\boldsymbol{R}T}{V_\mathrm{m} - b} - \frac{a}{T^{1/2}V_\mathrm{m}(V_\mathrm{m} + b)} = \frac{\rho\boldsymbol{R}T}{1 - b\rho} - \frac{a\rho^2}{\sqrt{T}(1 + b\rho)}$$

- Peng – Robinson EOS

$$\begin{aligned} p &= \frac{\boldsymbol{R}T}{V_\mathrm{m} - b} - \frac{a(T)}{V_\mathrm{m}(V_\mathrm{m} + b) + b(V_\mathrm{m} - b)} = \frac{\rho\boldsymbol{R}T}{1 - b\rho} - \frac{a\rho^2}{1 + b\rho + b\rho(1 - b\rho)} \\ z &= \frac{V_\mathrm{m}}{V_\mathrm{m} - b} - \frac{V_\mathrm{m}\,a(T)}{\boldsymbol{R}T[V_\mathrm{m}(V_\mathrm{m} + b) + b(V_\mathrm{m} - b)]} = \frac{1}{1 - b\rho} - \frac{a\rho}{\boldsymbol{R}T(1 + b\rho + b\rho(1 - b\rho))} \end{aligned} \tag{1.51}$$

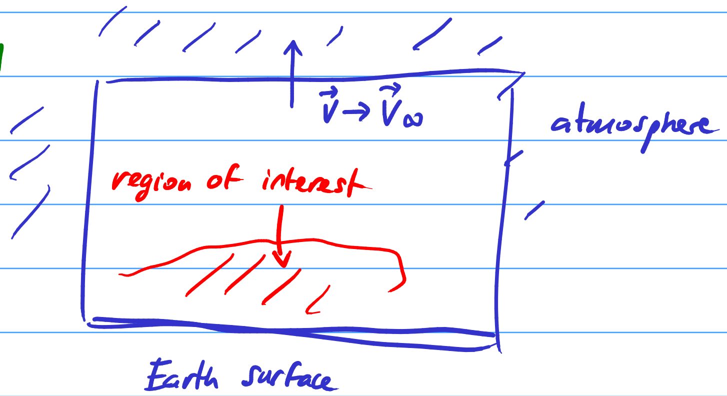
~~Parametry a, b jsou určeny relacemi~~ where

$$\begin{aligned} b &= 0.077\,796\,1\,\frac{\boldsymbol{R}T_\mathrm{c}}{p_\mathrm{c}}, \qquad a = \alpha \cdot a_\mathrm{c} = \alpha \cdot 0.457\,235\,52\,\frac{\boldsymbol{R}^2T_\mathrm{c}^2}{p_\mathrm{c}} \\ \alpha &= \left[1 + m\left(1 - \sqrt{T_\mathrm{r}}\right)\right]^2, \quad m = 0.374\,64 + 1.542\,26\,\omega - 0.269\,92\,\omega^2 \end{aligned} \tag{1.52}$$

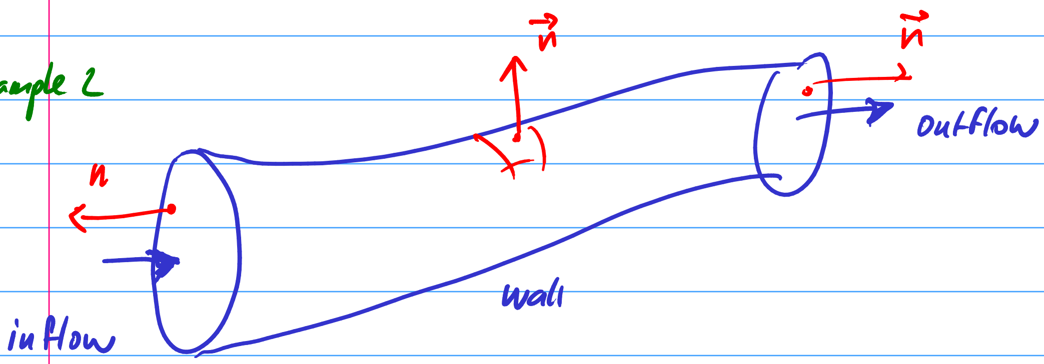
FORMULATION OF FLOW PROBLEMS, BOUNDARY CONDITIONS

– a domain Ω where the conservation laws are satisfied

Example 1



Example 2



how to prescribe boundary conditions (BC's)

- INVISCID FLOW – BC prescribed at the inflow only (!)

  $(\rho, T, \vec{V}_{in})$

at the outflow, the values are uniquely determined

at the wall : $\vec{V} \cdot \vec{n} = 0$ impermeability condition

2) viscous compressible flow

$$\text{at} \begin{cases} \text{inflow} : \vec{V}, T \\ \text{outflow} : P(\rho) \end{cases} \quad \Big| \quad \begin{matrix} P, T \\ \vec{V} \end{matrix} \quad \Big| \quad \begin{matrix} P_{in}, T \\ P_{out} \end{matrix}$$

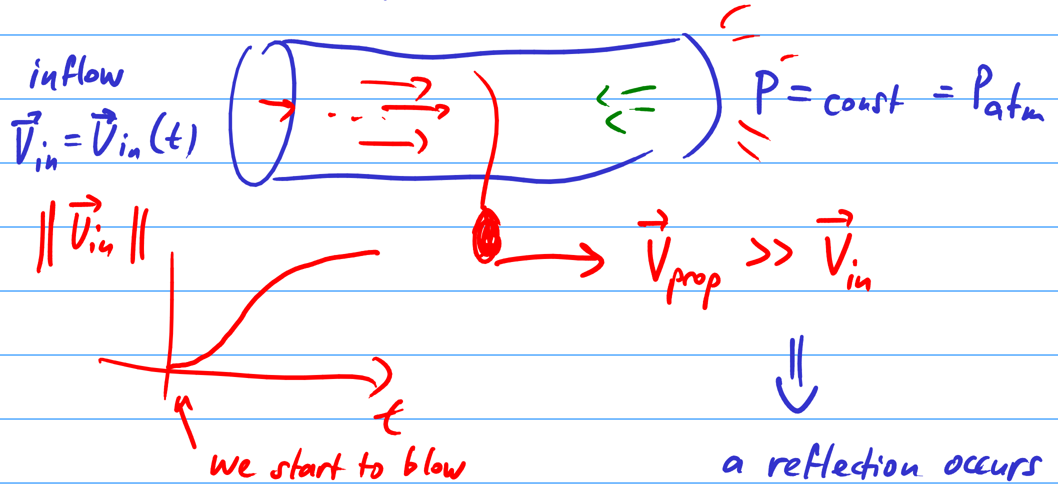
$P_{in} > P_{out}$

inflow $\Leftrightarrow$ $\vec{V} \cdot \vec{n} < 0$

outflow $\Leftrightarrow$ $\vec{V} \cdot \vec{n} > 0$ ... but this may change during the simulation

on the wall: $\vec{V} = \vec{0}$ (no-slip B.C.)

<u>REMARK:</u> non-stationary effects



- any f, for which B.C. is not explicitly prescribed, satisfies $\dfrac{\partial f}{\partial \vec{n}} = 0$

- reflection suppression
  - "resistance boundary condition"
  - B.C. based on Fourier expansions of $\vec{V}$

- periodic boundary conditions
(turbomachinery applications)